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Tunneling of the blocked wave in a circular hydraulic jump

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ABSTRACT

The formation of a circular hydraulic jump in a thin liquid layer involves the creation of a horizon where the incoming wave (surface ripples) is blocked by the fast flowing fluid. That there is a jump at the horizon is due to the viscosity of the fluid which is not relevant for the horizon formation. By using a tunneling formalism developed for the study of the Hawking radiation from black holes, we explicitly show that there will be an exponentially small tunneling of the blocked wave across the horizons as anticipated in studies of "analog gravity".

1. Introduction

A jet of fluid impinging vertically at high speed on a flat surface initially flows out radially in a thin almost laminar layer. At a certain critical radius it jumps to a thicker layer and the flow becomes turbulent. For a high viscosity, low surface tension liquid the jump is often along a circular ring and is called a circular hydraulic jump [1–4]. There are two essential ingredients in the jump:

- The formation of a circular boundary at the critical radius $r = R_c$, which is a demarcation between one-way and two-way flow of waves.
- The formation of the jump itself due to the presence of viscosity and the one-way information flow resulting in a bottleneck at the boundary.

Below, we see each of these in slightly greater detail.

Assuming the flow to be wholly radial, the formation of the boundary at $r = R_c$ is obtained from the condition

$$v_0(R_c) = c(R_c), \tag{1}$$

where $v_0(r)$ is the flow velocity of the water from the ground frame, $c(r)$ is the wave velocity of the ripples (which are the relevant hydrodynamic waves in a thin fluid) in the flowing frame, and $c^2 = gh_0$ where $h_0(r)$ is the height of the fluid layer. The height $h_0(r)$ and hence c are in general slowly varying functions of r. This condition has nothing to do with viscosity and gives a boundary but not a jump. The continuity equation for the flow requires the volumetric flow rate f to be a constant Q independent of the radial position r

$$f = r v_0(r) h_0(r) = Q\,. \tag{2}$$

The planar radial flow starting from the origin (the point where the jet impinges) is very fast with $v_0(r)$ initially greater than the ripple velocity $c(r)$. The thin fluid layer height $h_0(r)$ is slowly varying and from Eq. (2) we note that $v_0(r)$ has to decrease as the flow spreads out. At $r = R_c$ it equals $c(r)$. The radius R_c has a special significance. To see that let us consider the velocity of the ripple as seen by a laboratory observer. The result $c^2 = gh$ is valid for an observer for whom the fluid is stationary. A positive c is an outgoing wave and a negative c is an incoming wave. If the fluid is now moving with a velocity v_0 to the right with respect to the stationary observer (laboratory frame) then the velocity of the ripple according to that observer is $v_0 + c$. If $r > R_c$, $v_0 + c > 0$ for $c > 0$ and $v_0 + c < 0$ for $c < 0$. Hence the laboratory observer concludes that for $r > R_c$, there is both an outgoing and incoming wave. On the other hand if $r < R_c$, then for both positive and negative c, $v + c > 0$ and hence there is NO incoming wave for the laboratory observer. This is the phenomenon of wave blocking – the incoming wave is blocked from entering the region $r < R_c$. The radius $r = R_c$ is a horizon in the sense that in the interior there is only an outgoing wave and in the exterior there are both outgoing and incoming waves. This horizon is the reverse of a black hole horizon (only incoming waves for $r > R_c$) and is usually referred to as a white hole [5–7].

The formation of the jump at the boundary is the handiwork of viscosity [8–13]. Viscosity slows down the fluid layers as they move downstream and this affects the height of the layer. The effect is most extreme at the boundary itself. There, the sudden vanishing of incoming waves prevents the information about vis-

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cous retardation from being propagated towards the origin, causing a bottleneck and a resulting prominent increase in height. This is the jump itself. This phenomenon was captured in the WKB type of analysis in Ref. [13].

Assuming the above picture is correct, one can obtain the scaling law for the jump radius R_c. This is done by equating two time scales – the time scale τ_A corresponding to the time required to flow from the origin to the horizon and the viscous time scale τ_B. The respective estimates are $\tau_A \simeq R_C/v_0 = R_C/(gh_0)^{1/2}$ and $\tau_B = h_0^2/\eta$ where η is the kinematic viscosity. The volumetric flow rate evaluated at the horizon gives $Q_C = R_C g^{1/2} h_0^{3/2}$. Now setting $\tau_A = \tau_B$ as the jump condition, we arrive at

$$R_c \sim Q^{5/8} \nu^{-3/8} g^{1/8}, \tag{3}$$

which is a well known scaling law in this field [8]. Further refinements pertain to the inclusion of surface tension effects [14] and the formation of polygonal jumps [15,16]. We do not consider the effect of viscosity any more in this work except to show in the paragraph below that when the jump occurs at the horizon, the correct scaling of the jump radius follows from the qualitative picture given here. Our concern in what follows thereafter is with the horizon formation (wave blocking) alone.

The horizon formation explained above led to a whole body of extremely interesting work pointing out the possibility of an analogue of "Hawking radiation" in such cases [17,18]. This essentially means that it should be possible for some incoming wave to tunnel across the horizon to $r < R_C$. In the case of the real Hawking radiation the emission from a black hole is due to quantum fluctuations and quantum field theory is the proper setting for its study. However, it was realized that a tunneling formalism [19,20] for the quantum fluctuations could also be set up and though there were some quantitative issues at first, the correctness of the tunneling picture was finally established by the work of Partha Mitra and co-workers [21–23].

The most comprehensive discussion of the "Hawking effect" in analogue gravity from the view point of quantum field theory has been that of Scott Robertson [24]. In the hydraulic jump situation, the phenomena involved are completely classical, so it should be possible to view it as a linear stability analysis around the static solution. This latter solution gives the picture of wave blocking and the fluctuations around this steady state could appear as the tunneling that has been studied by Mitra [21]. In this Letter I show that such an approach is indeed possible, and obtain the presence of an exponentially decaying "wrong-way" wave beyond the horizon together with an estimate of its amplitude.

2. The calculation

In the black hole problem, one looks at quantum fluctuations around a Schwarzschild background. Here we begin with a static solution corresponding to $v_0(r)$ and $h_0(r)$ and consider small fluctuations $\delta v(r,t)$ and $\delta h(r,t)$ around them. This causes the volumetric flow rate f in Eq. (2) to fluctuate and we denote that fluctuation by f'. Linearizing the hydrodynamic equations around the steady state leads to the well known result [13]

$$\partial_\alpha \left(g^{\alpha\beta} \partial_\beta f' \right) = 0 \tag{4}$$

with

$$g^{\alpha\beta} = v_0 \begin{bmatrix} 1 & v_0 \\ v_0 & v_0^2 - gh_0 \end{bmatrix} \tag{5}$$

The metric element g^{22} has a zero at R_c where the flow velocity and wave velocity match. The wave blocking that this causes was explained physically before. We now point out that this metric corresponds to the one describing a Painleve–Gullstrand line element and this is in turn related to the Schwarzschild metric as explained in detail in Barcelo et al. [25]. This is the quantitative justification for the existence of a horizon at R_c. The vanishing of g^{22} at $r = R_c$ implies the existence of a horizon. We write out the equation for f' as

$$v_0 \partial_t^2 f' + v_0^2 \partial_t \partial_r f' + \partial_r \left(v_0^2 \partial_t f' \right) + \partial_r \left(v_0 \left(v_0^2 - c^2 \right) \partial_r f'' \right) = 0 \tag{6}$$

We look for separable solutions $f'(r,t) = S(r)\psi(t)$, substitute this ansatz into the above and find that

$$v_0 S \frac{\ddot{\psi}}{\psi} + \partial_r \left(v_0^2 S \right) \frac{\dot{\psi}}{\psi} + v_0^2 \partial_r S \frac{\dot{\psi}}{\psi} + \partial_r \left(v_0 \left(v_0^2 - c^2 \right) \partial_r S \right) = 0 \tag{7}$$

Separation of variables now requires that $\dot{\psi}/\psi = const$. Without loss of generality we can write the constant as $-\mathrm{i}\omega$, with the spectrum of ω being continuous. Then $\psi = \mathrm{e}^{-\mathrm{i}\omega t}$ with the constant of integration absorbed into S for now. Solving Eq. (7) with this ansatz yields the spatial part $S(r,\omega)$. Adding up all the separable solutions for each ω yields that

$$f'(r,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \mathrm{e}^{-\mathrm{i}\omega t} \, S(r,\omega), \tag{8}$$

i.e. $f'(r,t)$ is the Fourier transform of $S(r,\omega)$. The spatial part $S(r,\omega)$ satisfies

$$\left(v_0^2 - c^2 \right) \frac{\mathrm{d}^2 S}{\mathrm{d}r^2} + \left[3 v_0 \frac{\mathrm{d}v_0}{\mathrm{d}r} - \frac{1}{v_0} \frac{\mathrm{d}}{\mathrm{d}r} \left(v_0 c^2 \right) - \mathrm{i} 2 v_0 \omega \right] \frac{\mathrm{d}S}{\mathrm{d}r} - \left(\omega^2 + \mathrm{i} 2\omega \frac{\mathrm{d}v_0}{\mathrm{d}r} \right) S = 0 \tag{9}$$

Since we are looking for a wave-like solution (the ripples) there is a definite expectation about the form of $S(r,\omega)$. The ripples have wavelength much smaller than R_c and hence we are looking at solutions with high wave-number and consequently high frequency. Near the horizon, $v_0 \simeq c$ and at high frequencies Eq. (9) admits WKB type treatment. This was first noted in an astrophysical context by Petterson et. al. [26]. Accordingly we expand

$$S(r,\omega) = \exp \mathrm{i} K_\omega(r) = \exp \mathrm{i} \left(\sum_{n=-1}^{\infty} \frac{\varphi_n(r)}{\omega^n} \right) \tag{10}$$

where $\varphi_n(r)$ are complex-valued functions of r. The leading term $\varphi_{-1}(r)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}r} \varphi_{-1}(r) = \frac{1}{v_0 \pm c}. \tag{11}$$

In the above expression, we have, for the positive sign, and outgoing wave for both $r < R_c$ and $r > R_c$. However when we take the negative sign there is an incoming wave for $r > R_c$ and again an outgoing wave for $r < R_c$. This is the wave blocking we discussed before, now seen quantitatively.

At the next order i.e. $n = 0$ we get

$$\varphi_0 = \mathrm{i} \frac{1}{2} \log(c v_0) + \text{const}. \tag{12}$$

For two significant terms the solution is

$$S(r,\omega) = \frac{1}{(c v_0)^{1/2}} \left[A_1 \exp \mathrm{i}\omega \left(\int \frac{\mathrm{d}r}{v_0 + c} \right) + A_2 \exp \mathrm{i}\omega \left(\int \frac{\mathrm{d}r}{v_0 - c} \right) \right] \tag{13}$$

where A_1 and A_2 are constants of integration. For large r, $\varphi_{-1} \sim r$ while $\varphi_0 \sim \log r$, which is an indicator of successive terms falling

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