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From ordinary to discrete quantum mechanics: The Charlier oscillator and its coalgebra symmetry

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ABSTRACT

The coalgebraic structure of the harmonic oscillator is used to underline possible connections between continuous and discrete superintegrable models which can be described in terms of SUSY discrete quantum mechanics. A set of 1-parameter algebraic transformations is introduced in order to generate a discrete representation for the coalgebraic harmonic oscillator. This set of transformations is shown to play a role in the generalization of classical orthogonal polynomials to the realm of discrete orthogonal polynomials in the Askey scheme. As an explicit example the connection between Hermite and Charlier oscillators, that share the same coalgebraic structure, is presented and a two-dimensional maximally superintegrable version of the Charlier oscillator is constructed.

1. Introduction

A superintegrable system is, roughly speaking, a n-dimensional Hamiltonian system that allows a number $m > n$ of integrals of motion. If $m = 2n - 1$ then the system is called *Maximally Superintegrable* (MS). Due to its high number of symmetries, this class of systems is of great importance in mathematical physics. On the one hand, MS systems find many applications as exactly solvable models [1] in several areas of physics, such as condensed matter physics, nuclear physics and celestial mechanics [2–7]. On the other hand, they are also of interest in pure mathematics, due to the multiple connections with group theory and the search for new classes of orthogonal polynomials, just to cite a few (see the review paper [8] and references therein). The search and classification for MS systems has been performed over the years by using many different approaches. A possibility consists in considering a general Hamiltonian $H(\mathbf{x}, \mathbf{p}) = \mathbf{p}^2 + V(\mathbf{x})$ and imposing the existence of a set of constants of the motion $\{I_i\}$ under some specific assumptions such that $\{H, I_i\} = 0$ or $[H, I_i] = 0$, respectively in classical and quantum mechanics. The above constraints turn into a set of determining equations which can be used to classify completely a given class of MS systems. However, the complexity of these determining equations grows in a severe way proportionally to the Hamiltonian degrees of freedom. This partially justifies the abundance of studies of superintegrable systems in two dimensions [9–12]. In order to construct higher dimensional superintegrable systems without tackling the above mentioned issues, Ballesteros et al. introduced a novel algebraic approach based on coalgebras [13,14].

This technique consists in defining both the Hamiltonians and their constants of motion as functions of generators of a given algebra equipped with a coproduct. Once an algebra representation is chosen, then the coproduct can be used to rise the dimension of the representation without losing the superintegrability properties. This is because the coproduct provides, at each application, a set of additional symmetries "*the partial Casimirs*", which help to keep the system superintegrable. The coalgebra technique has been successfully used to construct new families of integrable and superintegrable systems, see e.g. [15,16].

As underlined in the previous paragraphs, two-dimensional MS systems have been object of deep studies and many classifications are available both in Euclidean and non-Euclidean spaces.

As recently showed, the coalgebraic analysis can also be used to give new insights about the already known classifications of superintegrable systems: in [17] it has been shown that a canonical transformation can generate different coalgebraic systems which, once embedded in higher dimensional spaces, generate genuinely new superintegrable systems as deformations, or generalizations, of TTW systems [18,19] to non-Euclidean spaces. The same philosophy has been reproposed in [20], involving a gauge transformation applied to a two-dimensional scalar Hamiltonian. In that case the two-dimensional coalgebraic Hamiltonian, when realized

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in three dimensions, turned out to be a non-scalar superintegrable Hamiltonian with spin interactions.

The main goal of the present paper is to carry on along this path, by showing that the coalgebraic structure could be successfully used to obtain discrete N-dimensional generalization of superintegrable families defined for continuous systems, or as an elegant way to unify different families of superintegrable systems as different realizations of a common coalgebraic structure. In particular, we focus the analysis on the prototype example of superintegrable system, the harmonic oscillator, whose discrete versions have been largely investigated both for their mathematical and physical interest (see [21–23] and the more recent papers [24,25]).

The paper is organized as follows:

- In section 2 we discuss the harmonic oscillator in terms of its coalgebra symmetry, both in the classical and the quantum case. This allows us to introduce the basic notions we need throughout the paper. Moreover, we review the spectral problem for the quantum harmonic oscillator in one and two dimensions in cartesian coordinates. For the latter case, the conserved quantities that make the system maximally superintegrable are reported.
- In section 3 we introduce an algebraic transformation that allows us to construct a discrete version of the $\mathfrak{sl}(2,\mathbb{R})$ coalgebra and, as a consequence, a discrete model of the quantum harmonic oscillator. We solve the spectral problem of the discrete oscillator model and, by using its coalgebra symmetry, we extend it to higher dimensions preserving the superintegrability of the Hamiltonian. For the sake of clarity, we focus to the two-dimensional case, for which the conserved quantities making the discrete system maximally superintegrable are explicitly constructed.
- In section 4 concluding remarks and future investigations are discussed.

2. The harmonic oscillator and the $\mathfrak{sl}(2,\mathbb{R})$ coalgebra

2.1. The classical case

Let us consider the following oscillator Hamiltonian (smooth) function $H:\mathcal{M}\to\mathbb{R}$, defined on the symplectic manifold ($\mathcal{M}=\mathbb{R}^2$, $\omega_0=dx\wedge dp$) (from now on we shall set $m=\omega=1$):

$$H(x,p)=\frac{p^2+x^2}{2}\,, \tag{1}$$

where x, p are canonical (local) coordinates on $\mathcal{M}$ such as $\{x,p\}=1$. This Hamiltonian can be expressed in terms of the generators of an $\mathfrak{sl}(2,\mathbb{R})$ coalgebra (see e.g. [26]):

$$\{J_-,J_+\}=4J_3\,,\ \{J_3,J_\pm\}=\pm 2J_\pm\,, \tag{2}$$

equipped with the (primitive) coproduct map $\Delta:\mathfrak{sl}(2,\mathbb{R})\to\mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R})$:

$$\Delta(J_\sigma)\doteq J_\sigma\otimes 1+1\otimes J_\sigma\ ,\ \ \Delta(1)\doteq 1\otimes 1\quad (\text{with } \sigma=\pm,3)\,. \tag{3}$$

A symplectic realization of (2) is given by:

$$D(J_+)=p^2\,,\ D(J_-)=x^2\,,\ D(J_3)=xp\,, \tag{4}$$

and the Casimir of the algebra is $C(J_\pm,J_3)\doteq J_+J_- - J_3^2=0$, in the given representation. In terms of these algebraic elements the Hamiltonian (1) can be expressed in the following form:

$$\begin{aligned}&(\mathfrak{sl}(2,\mathbb{R})\text{ algebra})\ H(J_-,J_+)=\frac{J_++J_-}{2}\\ &\xrightarrow{D} H(x,p)=\frac{D(J_+)+D(J_-)}{2}\ (\text{classical mechanics})\,.\end{aligned} \tag{5}$$

The advantage of having this representation for the Hamiltonian function is related to the mathematical properties of coalgebras. In particular, since the coproduct map defines a homomorphism for the Poisson algebra (2), i.e.:

$$\{\Delta(J_-),\Delta(J_+)\}=4\Delta(J_3)\ ,\quad \{\Delta(J_3),\Delta(J_\pm)\}=\pm 2\Delta(J_\pm)\,, \tag{6}$$

it is straightforward to extend the classical system to higher dimensions [13,14]. In fact, because of the homomorphism property, it is immediate to show that the two-particle Hamiltonian can be constructed through the coproduct of H, namely:

$$H^{(2)}\doteq\Delta(H(J_-,J_+))=H(\Delta(J_-),\Delta(J_+))\,, \tag{7}$$

and the new Hamiltonian will Poisson commute with the coproduct of the new Casimir $C^{(2)}\doteq\Delta(C)$, i.e.:

$$0=\Delta(\{H,C\})=\{\Delta(H),\Delta(C)\}=\{H^{(2)},C^{(2)}\}\,. \tag{8}$$

A crucial feature of the coproduct is related to the fact that it defines a *coassociative* map, which means that the following relation holds:

$$(\Delta\otimes\mathrm{id})\circ\Delta=(\mathrm{id}\otimes\Delta)\circ\Delta\,, \tag{9}$$

namely, the following diagram is commutative:

$$\begin{array}{ccc}\mathfrak{sl}(2,\mathbb{R}) & \xrightarrow{\ \ \Delta\ \ } & \mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R})\\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}}\\ \mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R}) & \xrightarrow{\mathrm{id}\otimes\Delta} & \mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R})\end{array}$$

Roughly speaking, this property allows one to define an object on $\mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R})$ in two possible ways, and this is intimately related to the *superintegrability* properties of the system under consideration. In fact, if we consider the more general N-particle realization of the $\mathfrak{sl}(2,\mathbb{R})$:

$$D(J_+^{(N)})=\sum_{k=1}^N p_k^2\ ,\ D(J_-^{(N)})=\sum_{k=1}^N x_k^2\ ,\ D(J_3^{(N)})=\sum_{k=1}^N x_kp_k\,, \tag{10}$$

where we defined $J_+^{(N)}\doteq\Delta^{(N)}(J_+)=\Delta(\Delta(\ldots(J_+)))$ (N-times), the coalgebra symmetry implies that the three algebra elements $J_{\pm,3}^{(N)}$ Poisson commute with a set of Casimir functions obtained by a "k-fold" ($2\le k\le N$) *left* or *right* application of the coproduct $\Delta:\mathfrak{sl}(2,\mathbb{R})\to\underbrace{\mathfrak{sl}(2,\mathbb{R})\otimes\mathfrak{sl}(2,\mathbb{R})\otimes\cdots\otimes\mathfrak{sl}(2,\mathbb{R})}_{N\text{-times}}$, i.e.:

$$\{J_{\pm,3}^{(N)},C^{(k)}\}=\{J_{\pm,3}^{(N)},C_{(k)}\}=0\,, \tag{11}$$

$$\begin{cases}C^{(k)}\doteq\Delta^{(k)}(C)\underbrace{\otimes 1\otimes\cdots\otimes 1}_{N-k}\,,\\ C_{(k)}\doteq\underbrace{1\otimes\cdots\otimes 1}_{N-k}\otimes\Delta^{(k)}(C)\,.\end{cases} \tag{12}$$

Since $C^{(N)}=C_{(N)}$, we will have a total number of $2N-3$ Casimirs $C^{(2)}\ldots C^{(N)},C_{(2)}\ldots C_{(N-1)}$ that, due to the above property, are in involution with the N-dimensional Hamiltonian[1]:

$$H^{(N)}(J_-^{(N)},J_+^{(N)})=\frac{J_+^{(N)}+J_-^{(N)}}{2}=\sum_{j=1}^N\frac{p_j^2+x_j^2}{2}\,. \tag{13}$$

This allows us to conclude that the system we are dealing with is, by construction, *quasi maximally superintegrable* (*QMS*), since it

[1] In the following, to simplify the notation, we will often drop the symbol D to indicate the given representation.