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Geometric phases in discrete dynamical systems

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ABSTRACT

In order to study the behaviour of discrete dynamical systems under adiabatic cyclic variations of their parameters, we consider discrete versions of adiabatically-rotated rotators. Parallelling the studies in continuous systems, we generalize the concept of geometric phase to discrete dynamics and investigate its presence in these rotators. For the rotated sine circle map, we demonstrate an analytical relationship between the geometric phase and the rotation number of the system. For the discrete version of the rotated rotator considered by Berry, the rotated standard map, we further explore this connection as well as the role of the geometric phase at the onset of chaos. Further into the chaotic regime, we show that the geometric phase is also related to the diffusive behaviour of the dynamical variables and the Lyapunov exponent.

1. Introduction

In continuous time dynamics, the study of adiabatic perturbations in general, and of adiabatic cyclic variations in particular, is closely related to the concepts of anholonomy and geometric phase. The geometric phase [1,2] is, indeed, a particular example of anholonomy that we can phrase as the failure of certain variables to return to their original values after a closed circuit in the parameters. Physical expressions of such anholonomies appear in the rotation of the plane of oscillation of a Foucault pendulum [3], how swimming is performed by microorganisms at low Reynolds numbers [4], how the stomach mixes [5], and how a falling cat can manage to reorientate itself in mid air in order to land on its feet [6]. The geometric phase was originally encountered — as Berry's phase — in quantum mechanics [7,8]. From there, it was generalized to classical integrable systems as Hannay's angle [9]. Later it was extended to nonintegrable perturbations of Hamiltonian systems [10–12], and thence to dissipative systems [13–15], all these instances within the context of continuous-time dynamics. In the same context, rotated rotators have been natural models in which to study this phenomenon because they provide an easy way to control the adiabatic nature of the cyclic variation of the parameter. With this in mind, Berry and Morgan, for instance, investigated the geometric phase of a continuous-time Hamiltonian rotated rotator [16]. In spite of the extensive research that has taken place in the last few decades on geometric phases in a large class of applications, neither the geometric phase nor any of its cognates have been considered hitherto in discrete dynamical systems. Moreover, the general question of how a mapping-defined dynamics behaves under an adiabatic parametric cyclic perturbation has not been addressed until now. Our purpose in this paper is to make good this deficit and to introduce a discrete analogue of the geometric phase and show that it is linked to important aspects of the dynamics of maps. In order to do so, we adopt the paradigm of the rotated rotator but follow an inverse sequence to the historical development. In the first place we deal with the sine circle map that may be thought of as the discretization of a kicked rotator of the type Berry and Morgan studied with the addition of strong dissipation. We consider the results of discrete adiabatically-evolving parameter loops in such a prototypical discrete-time dynamical system. The geometric phase in the rotated circle map, it turns out, is intimately related to the behaviour of the rotation number of the map as a function of the bare frequency parameter. Turning to the Hamiltonian side, we study the rotated standard map, in which we discover surprising relationships between the geometric phase, not

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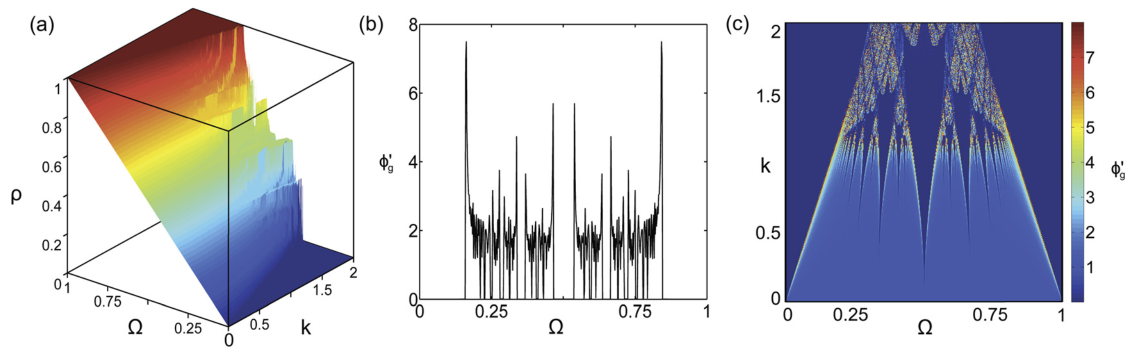

Fig. 1. (a) Devil's quarry plot of rotation number ρ in the sine circle map; a section through the quarry with k constant is a devil's staircase. (b) Geometric phase ϕ_g plotted against Ω for the rotated critical ($k = 1$) sine circle map of Eq. (4). (c) In a colormap, same as in (b) but as a function of both parameters Ω and k.

only with the rotation number as in the former case, but also with the Lyapunov exponent and the diffusive behaviour of both action and phase variables. The reason for our following this inverted developmental sequence is that 1D circle maps are simpler as regards the transition from integrability to chaos, in comparison with the much richer behaviour of the 2D Hamiltonian case.

2. The rotated circle map

Before introducing our rotated version of the circle map, let us first recall a few necessary definitions and results on the original non-rotated one. The circle map, usually written as

$$\theta_{n+1} = f^{n+1}_{\Omega,k}(\theta_0) = \theta_n + \Omega - \frac{k}{2\pi}\sin 2\pi\theta_n \mod 1, \tag{1}$$

where $\theta_n = f^n_{\Omega,k}(\theta_0)$ represents the nth iterate of θ_0, which qualitatively describes the dynamics of two interacting nonlinear oscillators, is a one-dimensional discrete mapping that describes how a rotator of natural frequency Ω behaves when forced at frequency one through a coupling of strength k. When $k = 0$ the rotator runs uncoupled at frequency Ω, but when $k > 0$ it can lock into a periodic orbit: a resonance with some rational ratio p/q to the driving frequency. To measure the frequency of the rotator, i.e., the average rotation per iteration of the map, it is useful to define the rotation number

$$\rho = \lim_{N\to\infty} \frac{\theta_N - \theta_0}{N}. \tag{2}$$

If we plot rotation number ρ against Ω and k — a few hundred iterations after discarding an initial transient are sufficient to give an accurate value for ρ — we obtain the devil's quarry [17] illustrated in Fig. 1(a). Periodic orbits with different rational rotation numbers show up as so-called Arnold tongues: flat steps in Fig. 1(a). When $k < 1$ the map is termed subcritical, and intervals on which the rotation number is constant and rational, where there is a periodic orbit of a particular period, punctuate intervals of increasing rotation number, whereas in supercritical circle maps ($k > 1$) the periodic orbits overlap. Chaos is found in the supercritical circle map as iterates wander between the overlapping resonances. In a critical circle map at $k = 1$, at every value of Ω there is a periodic orbit, and the rotation number increases in a staircase fashion with steps at each rational rotation number and risers in between. The devil's quarry becomes the devil's staircase when we look at a section with k constant through the quarry. The ordering of periodic orbits in the devil's staircase has been understood in terms of Farey sequences and Stern–Brocot trees [18–21], and the transition to chaos in the circle map is well understood.

Now let us consider the rotated circle map; probably the simplest discrete time system with a discretely and adiabatically varying parameter. To this end we introduce a discrete slowly varying parameter X_n:

$$\begin{aligned}\theta_{n+1} &= \theta_n + \Omega - \frac{k}{2\pi}\sin 2\pi(\theta_n + X_n) \mod 1,\\ X_{n+1} &= X_n + \Delta X = X_n \pm 1/N,\end{aligned} \tag{3}$$

where $n = 1, 2, \ldots N$, with $N \to \infty$ for adiabaticity. Is there a geometric contribution to the phase after such an excursion? Let us perform the change of variable $\theta'_n = \theta_n + X_n$, under which the map can be written as

$$\begin{aligned}\theta'_{n+1} &= \theta'_n + (\Omega + \Delta X) - \frac{k}{2\pi}\sin 2\pi\theta'_n \mod 1\\ &= \theta'_n + \Omega' - \frac{k}{2\pi}\sin 2\pi\theta'_n \mod 1,\end{aligned} \tag{4}$$

where $\Omega' = \Omega \pm 1/N$. So it is seen that the effect of the parameter loop is just a shift in the value of Ω.

In general, if one takes a system through a parameter loop, one obtains as a result three phases: a dynamic phase, a nonadiabatic phase, and a geometric phase. If one then traverses the same loop in the opposite direction, the dynamic phase accumulates as before, while the geometric phase is reversed in sign. There is, of course, still the nonadiabatic phase too; to get rid of this one must travel slowly around the loop. Thence the geometric phase may be obtained as

$$\phi_g = \lim_{N\to\infty} \frac{\phi_+ - \phi_-}{2}, \tag{5}$$

where ϕ_+ and ϕ_- are, respectively, the total angles θ accumulated by travelling around the loop in the positive and negative directions. In terms of the primed variables, we can also define

$$\phi_g{}' = \lim_{N\to\infty} \frac{f^N_{\Omega+1/N,k}(\theta_0) - f^N_{\Omega-1/N,k}(\theta_0)}{2}, \tag{6}$$

the obvious relation $\phi_g = \phi_g{}' - 1$. Let us evaluate this limit, first for the simple case $k = 0$. Then $f^N_{\Omega,0} = \theta_0 + N\Omega$, so $f^N_{\Omega-1/N,0} = \theta_0 + N\Omega - 1$ and $f^N_{\Omega+1/N,0} = \theta_0 + N\Omega + 1$, hence $\phi_g{}' = 1$ and $\phi_g = 0$. This limiting case is conceptually equivalent to that of a Foucault pendulum located at the Earth's equator where the plane of oscillations remains fixed as the Earth rotates.

More interesting is what happens when $k \neq 0$. From the definition of the rotation number

$$\rho = \lim_{N\to\infty} \frac{f^N_{\Omega,k}(\theta_0) - \theta_0}{N} = \lim_{N\to\infty} \rho^N_{\Omega,k}(\theta_0), \tag{7}$$

where we are defining $\rho^N_{\Omega,k}(\theta_0) = (f^N_{\Omega,k}(\theta_0) - \theta_0)/N$, we have that

$$\begin{aligned}\phi_g{}' &= \lim_{N\to\infty} \frac{f^N_{\Omega+1/N,k}(\theta_0) - f^N_{\Omega-1/N,k}(\theta_0)}{2}\\ &= \lim_{N\to\infty} \frac{\rho^N_{\Omega+1/N,k}(\theta_0) - \rho^N_{\Omega-1/N,k}(\theta_0)}{2/N}.\end{aligned} \tag{8}$$