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Parameter-free resolution of the superposition of stochastic signals

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ABSTRACT

This paper presents a direct method to obtain the deterministic and stochastic contribution of the sum of two independent stochastic processes, one of which is an Ornstein–Uhlenbeck process and the other a general (non-linear) Langevin process. The method is able to distinguish between the stochastic processes, retrieving their corresponding stochastic evolution equations. This framework is based on a recent approach for the analysis of multidimensional Langevin-type stochastic processes in the presence of strong measurement (or observational) noise, which is here extended to impose neither constraints nor parameters and extract all coefficients directly from the empirical data sets. Using synthetic data, it is shown that the method yields satisfactory results.

1. Introduction and motivation

An important topic in the analysis of time-series of complex dynamical systems is the extraction of the underlying process dynamics. Often it is possible to reveal the deterministic and stochastic contributions of the underlying stochastic process using the Itô–Langevin equation, a stochastic equation that describes the evolution of a stochastic variable. The deterministic and stochastic contributions are given by the so-called drift and diffusion coefficients, which can be directly derived from data via joint moments [4]. This approach has been applied successfully to several areas [5], for example the description of turbulence [4,17], the analysis of climate data [10], financial data [16], biological systems [19] and wind energy production [15,13].

However, typically the time-series to be analyzed is subject to noise, which is associated to the measurement devices or other sources. This so-called measurement noise, also known as observational noise, is not involved in the dynamics of the original signal. Nevertheless it spoils the data series by hiding the underlying stochastic process. In this case, the joint moments are not accessible but only their "noisy" analogues. Several approaches have been published to overcome this challenge. The authors of Refs. [3] and [9] introduced a method that allows the estimation of the drift and diffusion coefficients in the presence of strong, delta-correlated Gaussian measurement noise. An alternative approach was presented by Lehle [7] that can deal with strong, exponentially correlated Gaussian noise in one dimension, which was extended to be applicable to multidimensional time-series [8]. This approach is the basis of the method presented in this paper. Here, instead of using a parameterized form of the coefficients defining the stochastic processes, the method extracts all coefficients directly from the data.

In a more general framework, the paper presents a method which allows to distinguish between two superposed signals, i.e. extract their respective evolution equations, if one of them is an Ornstein–Uhlenbeck process. Specifically, the method serves to extract the measurement noise parameters as well as the drift and diffusion coefficients describing the stochastic process from the original data, which henceforth are called "noisy" data. This allows to separate the two stochastic signals: the measurement noise, described by an Ornstein–Uhlenbeck process, and the underlying general Langevin process. The method can be applied to a set of N coupled stochastic variables superposed with a set of N sources of correlated measurement noise and the code is accessible by request to the authors.

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The paper is structured as follows. The theoretical background of the Langevin analysis of stochastic processes and the extraction of the coefficients from data is briefly summarized in Sec. 2. Section 3 gives an overview of the method to obtain those coefficients in the presence of measurement noise. Subsequently, the two main challenges in the method are presented: a) the solution of a nonlinear equation system to obtain the measurement noise parameters, which is described in Sec. 4, and b) the solution of a system of convolution equations to estimate the joint moments of the underlying stochastic process, which is described in Sec. 5. The results of application to a synthetic data set are shown in Sec. 6, demonstrating the accuracy of the presented approach as well as its limits. Section 7 discusses possible applications of the method and concludes the paper.

2. A general model for noisy stochastic processes

The evolution of a stochastic variable can be described by the Itô–Langevin equation, a stochastic equation defined by a deterministic contribution (drift) and fluctuations from possible stochastic sources (diffusion). For the general case of a N-dimensional stochastic process $\mathbf{X}(t)$ the equation is given by:

$$d\mathbf{x} = \mathbf{D}^{(1)}(\mathbf{x})dt + \sqrt{\mathbf{D}^{(2)}(\mathbf{x})}d\mathbf{W}(t), \tag{1}$$

where $d\mathbf{W}$ denotes a vector of increments of independent Wiener processes with $\langle d\mathbf{W}_i\rangle = 0$ and $\langle d\mathbf{W}_i, d\mathbf{W}_j\rangle = \delta_{ij}dt\ \forall i, j = 1,\dots,N$, where $\langle\rangle$ denotes the average and δ_{ij} the Kronecker delta. Functions $\mathbf{D}^{(1)}(\mathbf{x})$ and $\mathbf{D}^{(2)}(\mathbf{x})$ are the Kramers–Moyal coefficients of the corresponding Fokker–Planck equation that describes the evolution of the conditional probability density function. In the case the distribution of initial conditions is known one can derive the evolution equation of the joint probability density function $f(\mathbf{x}, t)$ of the stochastic variables $\mathbf{x}$. It is given by:

$$\frac{\partial f(\mathbf{x},t)}{\partial t} = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left[D_i^{(1)}(\mathbf{x})f(\mathbf{x},t)\right] + \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_i\partial x_j}\left[D_{ij}^{(2)}(\mathbf{x})f(\mathbf{x},t)\right]. \tag{2}$$

The Kramers–Moyal coefficients, also called the drift ($\mathbf{D}^{(1)}(\mathbf{x})$) and diffusion ($\mathbf{D}^{(2)}(\mathbf{x})$) coefficients, can be directly derived from measurements [5]. However, here we consider that each measured stochastic variable is the sum of two independent stochastic processes X and Y:

$$\mathbf{X}^*(t) = \mathbf{X}(t) + \mathbf{Y}(t). \tag{3}$$

Since such a situation can be regarded as having a set of N stochastic signals $\mathbf{X}(t)$ spoiled by a set of N sources of measurement noise $\mathbf{Y}(t)$, we call the variables $\mathbf{X}^*(t)$ a N-dimensional *noisy* stochastic process. Fig. 1 shows a specific example of such superposition of stochastic processes that will be addressed below in detail, plotting the first component of $\mathbf{X}^*$, $\mathbf{X}$ and $\mathbf{Y}$. We assume the measurement noise $\mathbf{Y}(t)$ to be described by an Ornstein–Uhlenbeck process in N dimensions:

$$d\mathbf{y}(t) = -\mathbf{A}\mathbf{y}(t)dt + \sqrt{\mathbf{B}}d\mathbf{W}(t), \tag{4}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $N\times N$ matrices, $\mathbf{B}$ is symmetric positive semi-definite and the eigenvalues of $\mathbf{A}$ have a positive real part. Thus, the N-dimensional noisy stochastic process $\mathbf{X}^*$ is modeled by Eqs. (3) and (4) together. Note that here and throughout the paper $\mathbf{x}$ denotes the accessible values of any of the involved stochastic processes $\mathbf{X}(t)$, $\mathbf{X}^*(t)$ or $\mathbf{Y}(t)$.

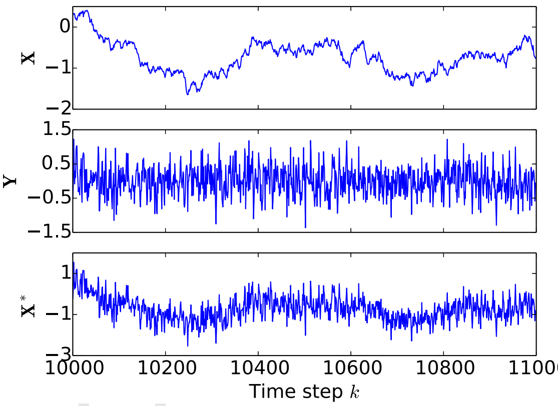

Fig. 1. Illustration of a stochastic process $\mathbf{X}(t)$ (top), governed by a nonlinear Langevin equation (Eq. (1)), a correlated measurement noise $\mathbf{Y}(t)$ (middle), governed by a Ornstein–Uhlenbeck process (Eq. (4)) and the superposition of both processes $\mathbf{X}^*(t) = \mathbf{X}(t) + \mathbf{Y}(t)$ (bottom).

3. From data to model: the inverse problem

This section explains how to obtain the drift and diffusion coefficients along with the measurement noise parameters from noisy data $\mathbf{X}^*(t)$. The methodology is sketched in Fig. 2 and the idea behind it is that, if the measurement noise is independent of the stochastic process, it is possible to derive an equation system that relates the noisy moments $m^{*(0)}(\mathbf{x}), \mathbf{m}^{*(1)}(\mathbf{x},\tau)$ and $\mathbf{m}^{*(2)}(\mathbf{x},\tau)$ with the measurement noise-free moments $m^{(0)}(\mathbf{x}), \mathbf{m}^{(1)}(\mathbf{x},\tau)$ and $\mathbf{m}^{(2)}(\mathbf{x},\tau)$ and solving it in a parameter-free way is the heart of this paper.

The system of equations (for a derivation see Appendix A) is given by:

$$m^{*(0)}(\mathbf{x}) \equiv \int_{\mathbf{x}'} \rho^*(\mathbf{x},\mathbf{x}',\tau)d\mathbf{x}' = \rho_Y(\mathbf{x}) * m^{(0)}(\mathbf{x}), \tag{5a}$$

$$m_i^{*(1)}(\mathbf{x},\tau) \equiv \int_{\mathbf{x}'} (x_i'(t+\tau) - x_i(t))\rho^*(\mathbf{x},\mathbf{x}',\tau)d\mathbf{x}' = \rho_Y(\mathbf{x}) * m_i^{(1)}(\mathbf{x},\tau) + H_i^{(1)}(\mathbf{A},\mathbf{B},m^{*(0)}(\mathbf{x})), \tag{5b}$$

$$m_{ij}^{*(2)}(\mathbf{x},\tau) \equiv \int_{\mathbf{x}'} (x_i'(t+\tau) - x_i(t))(x_j'(t+\tau) - x_j(t))\rho^*(\mathbf{x},\mathbf{x}',\tau)d\mathbf{x}' = \rho_Y(\mathbf{x}) * m_{ij}^{(2)}(\mathbf{x},\tau) + H_{ij}^{(2)}(\mathbf{A},\mathbf{B},m^{*(0)}(\mathbf{x}),\mathbf{m}^{*(1)}(\mathbf{x},\tau)), \tag{5c}$$

where $i, j = 1,\dots,N$ and

$$\rho^*(\mathbf{x}) = f(\mathbf{x},t), \tag{6a}$$

$$\rho^*(\mathbf{x},\mathbf{x}',\tau) = f(\mathbf{x},t;\mathbf{x}',t+\tau), \tag{6b}$$

are the one and two-point probability density functions of the noisy data, respectively, and

$$\rho_Y(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^N|\det(\mathbf{V})|}}e^{-\frac{1}{2}\mathbf{x}^T\mathbf{V}^{-1}\mathbf{x}}, \tag{7}$$

is the probability density function of the measurement noise, assuming that it is distributed with a normalized Gauss function $\mathcal{G}(\mathbf{x},0,\mathbf{V})$ with zero average and covariance $\mathbf{V}$. The functions $H_i^{(1)}(\mathbf{A},\mathbf{B},m^{*(0)}(\mathbf{x}))$ and $H_{ij}^{(2)}(\mathbf{A},\mathbf{B},m^{*(0)}(\mathbf{x}),\mathbf{m}^{*(1)}(\mathbf{x},\tau))$ are given by

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