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Optimal solutions for homogeneous and non-homogeneous equations arising in physics

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ABSTRACT

In this study, we present a new modified convergent analytical algorithm for the solution of nonlinear boundary value problems by the Variation of Parameters Method (VPM). The method is based on embedding, auxiliary parameter and auxiliary linear differential operator, provides a computational advantage for the convergence of approximate solutions for nonlinear differential equations. Convergence of developed scheme is also shown and discussed in detail. Moreover, a convenient way is considered for choosing an optimal value of auxiliary parameter via minimizing the residual error over the domain of problem. The accuracy and efficiency of the proposed algorithm is established by implementing on some physical problems. The obtained graphical and numerical results clearly reflects the accuracy and convergence of the presented algorithm.

Introduction

Homogenous and non-homogenous differential equations occur in various fields of engineering and physical sciences and have been briefly studied in the literature. The methods for solving Boundary value problems (BVPs) for differential equations have attracted a lot of attention. Dehghan and Tatari [1] proposed Adomian decomposition method (ADM) for obtaining an approximate solution of multi-point BVPs. Geng [2] used reproducing kernel Hilbert space method for second order three-point BVPs. Geng and Cui [3] presented the combination of Homotopy perturbation method and variational iteration method for solving nonlinear multi-point BVPs. Li and Wu [4] introduced a method for a class of linear singular fourth order four-point BVPs. Duan and Rach [5] proposed a new modification of the ADM for solving multi-point BVPs by inserting Daun's convergence parameter. Khalid et al. [6] used the Laplace transform technique to obtain exact solutions of partial differential equations for velocity and energy. Hussanan et al. [7] studied the heat transfer effect on the unsteady boundary layer flow of a casson fluid past an infinite oscillating vertical plates using the Laplace transform method. Qasim et al. [8] used the Runge-Kutta-Fehlberg fourth-fifth order for system of ordinary differential equation and examine the influence of different parameters on velocity, temperature and microrotation. Ilyas khan et al. [9–12] used the Laplace transform method to obtain closed form solution of ordinary differential equations representing non-Newtonian fluid flow. Abdul hameed et al. [13] investigated the exact solutions of stokes first and second problems using separation of variable method together with the similarity arguments.

We develop a new kind of analytical algorithm for multi-point BVPs by the Variation of Parameters Method (VPM) [14,15]. The VPM has been used to solve a wide class of nonlinear differential equations [16–25]. These publications focus on the approximate solution of nonlinear problems, particularly algorithm is applied to problems involving nonhomogeneous terms that is a function of dependent variable. The main advantage of method is that, it does not rely on linearization, discretization, perturbation and restrictive assumptions. The results obtained are completely reliable with the proposed algorithm and are very encouraging.

The contribution of present work is to implement Optimal Variation of Parameters Method (OVPM) for multi-point BVPs. The physical problems which we are consider are, Lane-Emden type equation, which describes a variety of physical phenomena such as thermal history of spherical cloud of a gas, aspects of staller structure, isothermal gas spheres and thermionic currents. Second order nonlinear convective- radiative-conduction equation which was studied in the application of one dimensional heat transfer in a straight fin. Fourth order nonlinear boundary value problem which represents the deformation of an elastic beam that at least one of its end is supported.

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According to OVPM, we consume all of the boundary conditions to establish an integral equation before constructing an iterative algorithm to obtain an approximate solution. Thus we establish a modified iterative algorithm that does not contain undetermined coefficients, whereas most previous iterative method do incorporate undetermined coefficients. It is observed that the coupling algorithm provide a convenient way to control and adjust the convergence region of approximate solution over the domain of the problem. The main advantage of developed algorithm is its ability in providing the better information of continuous approximate solution over the domain of the problem. In the modified algorithm the residual error is defined to choose an optimal value of auxiliary parameter. Convergence of modified scheme is also shown and discussed in detail. However, the convergence of different analytical schemes has already been proven by many researchers [26–29]. Four examples are given to explicitly reveal the performance and reliability of the suggested algorithm.

Optimal Variation of Parameters Method (OVPM)

To convey the basic step of OVPM for differential equations, we consider a general second order nonlinear ordinary differential equation in operator form as follows:

$$Lf(\eta) + Rf(\eta) + Nf(\eta) + g(\eta) = 0, \quad a \leqslant \eta \leqslant b, \tag{1}$$

subject to the boundary conditions

$$f(a) = \alpha, \quad f(b) = \beta, \tag{2}$$

where $L = \frac{d^2}{d\eta^2}$ represents the higher order linear operator, R shows a linear operator of order $Hf(\eta) = Hf(\eta) + \kappa(Lf(\eta) + Rf(\eta) + Nf(\eta) + g(\eta))$, less than L, N is a nonlinear operator, and g is an inhomogeneous term. An unknown auxiliary parameter κ can be coupled with Eq. (1), so that it can be rewritten in the following form:

$$Hf(\eta) = Hf(\eta) + \kappa(Lf(\eta) + Rf(\eta) + Nf(\eta) + g(\eta)), \tag{3}$$

where H is any suitable linear differential operator with coefficient of higher order derivative one. Here we consider $H = L = \frac{d^2}{d\eta^2}$, so that Eq. (3) becomes.

$$Lf(\eta) = Lf(\eta) + \kappa(Lf(\eta) + Rf(\eta) + Nf(\eta) + g(\eta)), \tag{4}$$

or

$$Lf(\eta) = Lf(\eta) + \kappa Gf(\eta). \tag{5}$$

According to Variation of Parameters Method [14,15,18,30], the complementary function of (5) is given as

$$f_c(\eta) = C_1 + C_2\eta, \tag{6}$$

for particular solution, the constants C_1 and C_2 in Eq. (6) are replaced by the functions $u_1(\eta)$ and $u_2(\eta)$ respectively, thus we have

$$f_p(\eta) = u_1(\eta)v_1(\eta) + u_2(\eta)v_2(\eta), \tag{7}$$

where $v_1(\eta) = 1$ and $v_2(\eta) = \eta$. The functions $u_1(\eta)$ and $u_2(\eta)$ can be determined as

$$u_1(\eta) = \int_0^\eta \frac{W_1}{W} d\zeta \quad \text{and} \quad u_2(\eta) = \int_0^\eta \frac{W_2}{W} d\zeta, \tag{8}$$

where W is the Wronskian of v_1 and v_2, i.e.

$$W = \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix} = 1, \tag{9}$$

$$W_1 = \begin{vmatrix} 0 & v_2 \\ Lf(\eta) + \kappa Gf(\eta) & v_2' \end{vmatrix} = -\eta(Lf(\eta) + \kappa Gf(\eta)) \tag{10}$$

and

$$W_2 = \begin{vmatrix} v_1 & 0 \\ v_1' & Lf(\eta) + \kappa Gf(\eta) \end{vmatrix} = Lf(\eta) + \kappa Gf(\eta). \tag{11}$$

Substituting the value of W, W_1 and W_2 in Eq. (8), yields

$$u_1(\eta) = -\int_0^\eta \zeta(Lf(\zeta) + \kappa Gf(\zeta))d\zeta \text{ and}$$
$$u_2(\eta) = \int_0^\eta (Lf(\zeta) + \kappa Gf(\zeta))d\zeta, \tag{12}$$

hence, the general solution of Eq. (5) is

$$\begin{aligned} f(\eta) &= f_c(\eta) + f_p(\eta) \\ &= C_1 + C_2\eta + \int_0^\eta (\eta - \zeta)(Lf(\zeta) + \kappa Gf(\zeta))d\zeta. \end{aligned} \tag{13}$$

Substituting $\eta = a$ and $\eta = b$ in Eq. (13) and solve for C_1 and C_2, we get

$$\begin{aligned} C_1 = \alpha - a\left(\frac{\alpha-\beta}{a-b}\right) &- \int_0^a (a-\zeta)(Lf(\zeta) + \kappa Gf(\zeta))d\zeta \\ &+ \frac{a}{a-b}\left(\int_0^a (a-\zeta)(Lf(\zeta) + \kappa Gf(\zeta))d\zeta \right. \\ &\left. - \int_0^b (b-\zeta)(Lf(\zeta) + \kappa Gf(\zeta))d\zeta\right), \end{aligned}$$

$$\begin{aligned} C_2 = \frac{\alpha-\beta}{a-b} - \frac{1}{a-b}&\left(\int_0^a (a-\zeta)(Lf(\zeta) + \kappa Gf(\zeta))d\zeta \right. \\ &\left. - \int_0^b (b-\zeta)(Lf(\zeta) + \kappa Gf(\zeta))d\zeta\right), \end{aligned}$$

substituting the value of C_1 and C_2 in Eq. (13) yields

$$\begin{aligned} f(\eta) = \alpha - a\left(\frac{\alpha-\beta}{a-b}\right) &+ \eta\left(\frac{\alpha-\beta}{a-b}\right) + \int_0^\eta (\eta-\zeta)(Lf(\zeta) \\ &+ \kappa(Lf(\zeta) + Rf(\zeta) + Nf(\zeta) + g(\zeta)))d\zeta \\ &+ \left(\frac{a-\eta}{a-b} - 1\right)\int_0^a (a-\zeta)(Lf(\zeta) + \kappa(Lf(\zeta) + Rf(\zeta) + Nf(\zeta) + g(\zeta)))d\zeta \\ &+ \frac{\eta-a}{a-b}\int_0^b (b-\zeta)(Lf(\zeta) + \kappa(Lf(\zeta) + Rf(\zeta) + Nf(\zeta) + g(\zeta)))d\zeta, \end{aligned} \tag{14}$$

which can be solved iteratively as

$$f_0(\eta) = \alpha - a\left(\frac{\alpha-\beta}{a-b}\right) + \eta\left(\frac{\alpha-\beta}{a-b}\right),$$

$$\begin{aligned} f_1(\eta,\kappa) = f_0(\eta) &+ \int_0^\eta (\eta-\zeta)\left(Lf_0(\zeta) + \kappa\begin{pmatrix} Lf_0(\zeta) + Rf_0(\zeta) \\ +Nf_0(\zeta) + g(\zeta) \end{pmatrix}\right)d\zeta \\ &+ \left(\frac{a-\eta}{a-b} - 1\right)\int_0^a (a-\zeta)\left(Lf_0(\zeta) + \kappa\begin{pmatrix} Lf_0(\zeta) + Rf_0(\zeta) \\ +Nf_0(\zeta) + g(\zeta) \end{pmatrix}\right)d\zeta \\ &+ \frac{\eta-a}{a-b}\int_0^b (b-\zeta)\left(Lf_0(\zeta) + \kappa\begin{pmatrix} Lf_0(\zeta) + Rf_0(\zeta) \\ +Nf_0(\zeta) + g(\zeta) \end{pmatrix}\right)d\zeta, \end{aligned} \tag{15}$$

for $n \geqslant 1$,

$$\begin{aligned} f_{n+1}(\eta,\kappa) = f_0(\eta) &+ \int_0^\eta (\eta-\zeta)\left(Lf_n(\zeta,\kappa) + \kappa\begin{pmatrix} Lf_n(\zeta,\kappa) + Rf_n(\zeta,\kappa) \\ +Nf_n(\zeta,\kappa) + g(\zeta) \end{pmatrix}\right)d\zeta \\ &+ \left(\frac{a-\eta}{a-b} - 1\right)\int_0^a (a-\zeta)\left(Lf_n(\zeta,\kappa) + \kappa\begin{pmatrix} Lf_n(\zeta,\kappa) + Rf_n(\zeta,\kappa) \\ +Nf_n(\zeta,\kappa) + g(\zeta) \end{pmatrix}\right)d\zeta \\ &+ \frac{\eta-a}{a-b}\int_0^b (b-\zeta)\left(Lf_n(\zeta,\kappa) + \kappa\begin{pmatrix} Lf_n(\zeta,\kappa) + Rf_n(\zeta,\kappa) \\ +Nf_n(\zeta,\kappa) + g(\zeta) \end{pmatrix}\right)d\zeta, \end{aligned} \tag{16}$$

Consequently, an exact solution may be achieved when n approaches to infinity: