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A novel approach to approximate fractional derivative with uncertain conditions

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ABSTRACT

This paper focuses on providing a new scheme to find the fuzzy approximate solution of fractional differential equations (FDEs) under uncertainty. The Caputo-type derivative base on the generalized Hukuhara differentiability is approximated by a linearization formula to reduce the corresponding uncertain FDE to an ODE under fuzzy concept. This new approach may positively affect on the computational cost and easily apply for the other types of uncertain fractional-order differential equation. The performed numerical simulations verify the proficiency of the presented scheme.

1. Introduction

In recent two decades, fractional differential equations (FDE) have caught a lot of attentions due to their inherent ability, which can depict many physical processes with inheritance or memory more accurately in various fields of science and engineering [1–4]. Many real world problems can be better modeled by fractional-order differential equations (DEs) rather than by integer-order DEs [5,6] and nowadays, it is not hard to find very interesting and novel applications of FDEs. In the meantime, the non-local feature brings about many challenges to the existing numerical methods in terms of memory storage, computational cost, accuracy, etc. One of the big questions arsing in the numerical simulations of FDEs is that how we can eliminate the fractional derivative from our computations to reduce the complexity and make a low cost and efficient method. Hence, there has been considerable interest in seeking numerical solutions of FDEs that describe some important physical and dynamic processes and are obtained with low computational costs [7–10].

Since the beginning of the fuzzy set theory in 1965, the mathematical advancements have progressed to exceptionally high qualities. Plenty of researches have been conducted on fuzzy systems and its implementations in many disciplines [11–15]. However, the fractional calculus base upon fuzzy concept was not investigated till 2010. In this year, Agarwal et al. [16] as the inventors exploring this interesting field, formulated the Riemann–Liouville (RL)-derivative with fuzzy notion. Followed by them, some researches were devoted to provide the theoretical foundations of this new area such as the existence and uniqueness of the solution, fuzzy Caputo derivative, fuzzy fractional functional differential equations, random fuzzy fractional integral equations and etc. [17–24].

The lack of numerical approach for the solution of FDEs under uncertainty, motivated a number of authors to develop some approximation techniques for solving this new type of FDEs [25–30]. From this perspective, in very recent years, demands for developing numerical methods have been attained a considerable attention because of the footprint of uncertainty in FDEs-based models can also be tuned to improve the performance of a real-world system. Similar to the crisp FDEs, the major issue here is to find some proficient numerical algorithms not only to reduce the complicated computations but also to have an acceptable accuracy. It is even more critical than non-fuzzy cases since we must solve two FDEs systems simultaneously in order to achieve the fuzzy approximate solution. This takes considerable complexity and needs more delicate studies. Thus, elegant approaches are highly desirable to find fuzzy approximate solutions with less efforts while the researches is not yet enough and satisfying.

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Motivated by the above discussion, in this study, a novel approach is presented to approximate the Caputo-type derivative under uncertainty. Indeed, a linearization formula is proposed to approximate fuzzy fractional derivative under fuzzy differentiability. Then, the FDE under uncertainty changes to a fuzzy ODE and the fractional derivative is omitted from the next step of computations. In this step, we can apply any numerical method implemented for fuzzy ODE to get the approximate solution. Here, we employ fuzzy Laplace transforms [31] to get the desired solution under both types of fuzzy differentiability. Moreover, a derivative theorem that was already presented in [26], is revisited to find the fuzzy exact solution for FDEs under uncertainty. This theorem provide a useful tool to obtain the exact solution via fuzzy Laplace transforms. We believe that this new scheme aid to establish the numerical simulations of the well-posedness of the real-world problems based on the uncertain FDEs.

The organization of this paper is as follows. In Section 2, some preliminary knowledge and results will be presented. This knowledge includes the definition of fuzzy numbers, fuzzy differentiability and fuzzy Caputo-type derivative. Then, the formulation of our approximation approach is proposed in Section 3 and the derivative theorem for the fuzzy exact solution based on Laplace transforms is also presented in this section. Numerical simulation results are reported in Section 4 to demonstrate the effectiveness of the given scheme arising in physical, engineering systems such as Basset problem. We conclude the paper in the last section.

2. Preliminaries and notation

First, let us present an overview of the significant properties of fuzzy settings and fractional calculus which are required for our scheme. Interested reader are referred to [3,4,11,32] and references there in.

Let $\mathcal{K}(\mathbb{R}^d)$ denote the collection of all nonempty compact and convex subsets of $\mathbb{R}^d$. and scalar multiplication in $\mathcal{K}(\mathbb{R}^d)$ as usual, i.e., for $A, B \in \mathcal{K}(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$,

$$A + B = \{a + b \mid a \in A, b \in B\}, \qquad \lambda A = \{\lambda a \mid a \in A\}.$$

The Hausdorff distance $\mathbf{D}$ in $\mathcal{K}(\mathbb{R}^d)$ is defined as follows:

$$\mathbf{D}(A, B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|_{R^d}, \sup_{b\in B}\inf_{a\in A}\|a-b\|_{R^d}\},$$

where $A, B \in \mathcal{K}(\mathbb{R}^d)$, $\|.\|_{R^d}$ denotes the Euclidean norm in $\mathbb{R}^d$. It is known that $(\mathcal{K}(\mathbb{R}^d), \mathbf{D})$ is complete, separable and locally compact. Define $\mathcal{F}^d = \{\omega : \mathbb{R}^d \to [0, 1]$ such that $\omega(z)$ satisfies (i)–(iv) stated below}:

(i) ω is normal, that is, there exists $z_0 \in \mathbb{R}^d$ such that $\omega(z_0) = 1$;
(ii) ω is fuzzy convex, that is, for $0 \le \lambda \le 1$,

$$\omega(\lambda z_1 + (1-\lambda)z_2) \ge \min\{\omega(z_1), \omega(z_2)\},$$

for any $z_1, z_2 \in \mathbb{R}^d$;
(iii) ω is upper semicontinuous;
(iv) $[\omega]^0 = cl\{z \in \mathbb{R}^d : \omega(z) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^d, \|\cdot\|)$.

The elements of $\mathcal{F}^d$ are often called the fuzzy numbers. For $\alpha \in (0, 1]$, define $[\omega]^r = \{z \in \mathbb{R}^d \mid \omega(z) \ge r\}$. We will call this set an r–cut (r– level set) of the fuzzy set ω. For $\omega \in \mathcal{F}^d$, one has that $[\omega]^r \in \mathcal{K}(\mathbb{R}^d)$ for every $r \in [0, 1]$. For two fuzzy sets $\omega_1, \omega_2 \in \mathcal{F}^d$, we denote $\omega_1 \le \omega_2$ if and only if $[\omega_1]^r \subset [\omega_2]^r$. If $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a function then, according to Zadeh's extension principle, one can extend g to $\mathcal{F}^d \times \mathcal{F}^d \to \mathcal{F}^d$ by the formula $g(\omega_1, \omega_2)(z) = \sup_{z=g(z_1,z_2)} \min\{\omega_1(z_1), \omega_2(z_2)\}$. It is well known that if g is continuous then $[g(\omega_1, \omega_2)]^r = g([\omega_1]^r, [\omega_2]^r)$ for all $\omega_1, \omega_2 \in \mathcal{F}^d, r \in [0, 1]$. Especially, for addition and scalar multiplication in fuzzy set space $\mathcal{F}^d$, we have $[\omega_1 + \omega_2]^r = [\omega_1]^r + [\omega_2]^r, [\lambda\omega_1]^r = \lambda[\omega_1]^r$. In the case $d = 1$, the r–cut set of a fuzzy number ω is a closed bounded interval $[\underline{\omega}(r), \overline{\omega}(r)]$, where $\underline{\omega}(r)$ denotes the left-hand endpoint of $[\omega]^r$ and $\overline{\omega}(r)$ denotes the right-hand endpoint of $[\omega]^r$. It should be noted that for $a \le b \le c$, $a, b, c \in \mathbb{R}$, a triangular fuzzy number $\omega = (a, b, c)$ is given such that $\underline{\omega}(r) = a + (b-a)r$ and $\overline{\omega}(r) = c - (c-b)r$ are the endpoints of the r–cut for all $r \in [0, 1]$. For $\omega \in \mathcal{F}^1$, we define the diameter of ω as $diam[\omega]^r = \overline{\omega}(r) - \underline{\omega}(r)$. Let us denote by

$$\mathbf{D}_0[\omega_1, \omega_2] = \sup_r\{\mathbf{D}([\omega_1]^r, [\omega_2]^r) : 0 \le r \le 1\}$$

the distance between ω_1 and ω_2 in $\mathcal{F}^d$, where $\mathbf{D}([\omega_1]^r, [\omega_2]^r)$ is Hausdorff distance between two set $[\omega_1]^r$, $[\omega_2]^r$ of $\mathcal{K}(\mathbb{R}^d)$. Then $(\mathcal{F}^d, \mathbf{D}_0)$ is a complete space. Some properties of metric $\mathbf{D}_0$ are as follows:

$$\mathbf{D}_0[\omega_1 + \omega_3, \omega_2 + \omega_3] = \mathbf{D}_0[\omega_1, \omega_2],$$
$$\mathbf{D}_0[\lambda\omega_1, \lambda\omega_2] = |\lambda|\mathbf{D}_0[\omega_1, \omega_2],$$
$$\mathbf{D}_0[\omega_1, \omega_2] \le \mathbf{D}_0[\omega_1, \omega_3] + \mathbf{D}_0[\omega_3, \omega_2],$$

for all $\omega_1, \omega_2, \omega_3 \in \mathcal{F}^d$ and $\lambda \in \mathbb{R}$. Let $\omega_1, \omega_2 \in \mathcal{F}^d$. If there exists $\omega_3 \in \mathcal{F}^d$ such that $\omega_1 = \omega_2 + \omega_3$, then ω_3 is called the H-difference of ω_1, ω_2 and it is denoted by $\omega_1 \ominus \omega_2$. Let us remark that $\omega_1 \ominus \omega_2 \ne \omega_1 + (-1)\omega_2$. Let us denote $\hat{0} \in \mathcal{F}^d$ the zero element of $\mathcal{F}^d$ as follows: $\hat{0}(z) = 1$ if $z = 0$ and $\hat{0}(z) = 0$ if $z \ne 0$, where 0 is the zero element of $\mathbb{R}^d$.

We define the space of continuous fuzzy functions as

$$C^{\mathcal{F}^d}([t_0, T]) = \{x : [t_0, T] \to \mathcal{F}^d \mid x \text{ is continuous}\},$$

which is a complete metric space endowed with the following metric

$$\mathbf{D}_0^*[x, \hat{x}] = \sup_{t\in[t_0,T]} \mathbf{D}_0[x(t), \hat{x}(t)], \text{ for } x, \hat{x} \in C^{\mathcal{F}^d}([t_0, T]).$$

In the following part, we review some main concepts and properties of fuzzy generalized H-differentiability for fuzzy functions, that was introduced in [33,34].

Definition 2.1 (See, [33]). Let $x : [t_0, T] \to \mathcal{F}^d$ and $t \in (t_0, T)$. We say that x is generalized differentiable at t, if there exists $\mathbf{D}_H^g x(t) \in \mathcal{F}^d$, such that either for all $h > 0$ sufficiently small, the generalized H-differences $x(t+h) \ominus_{gH} x(t), x(t) \ominus_{gH} x(t-h)$ exist and the limits (by the metric $\mathbf{D}_0$)

$$\lim_{h\searrow 0} \mathbf{D}_0\left[\frac{x(t+h)\ominus_{gH} x(t)}{h}, \mathbf{D}_H^g x(t)\right]$$
$$= \lim_{h\searrow 0} \mathbf{D}_0\left[\frac{x(t)\ominus_{gH} x(t-h)}{h}, \mathbf{D}_H^g x(t)\right] = 0$$

and a fuzzy function $\mathbf{D}_H^g x(t) \in \mathcal{F}^d$ is called a generalized H-derivative of fuzzy function $x(t)$.

Remark 2.1. If x is fuzzy generalized H-differentiable at t, then $x(t)$ may be FH^{g1}– differentiable or FH^{g2}– differentiable.

Lemma 2.1 *(See, [35]). Let $x(t) \in \mathcal{F}^1$ and the parametric form is assume as $[x(t)]^r = [\underline{x}(t, r), \overline{x}(t, r)]$ for each $r \in [0, 1]$.*

(i) *If $x(t)$ is (FH^{g1})-differentiable, then $\underline{x}(t, r), \overline{x}(t, r)$ are differentiable functions and we have $[\mathbf{D}_H^{g1} x(t)]^r = [\underline{x}'(t, r), \overline{x}'(t, r)]$.*
(ii) *If $x(t)$ is (FH^{g2})-differentiable, then $\underline{x}(t, r), \overline{x}(t, r)$ are differentiable functions and we have $[\mathbf{D}_H^{g2} x(t)]^r = [\overline{x}'(t, r), \underline{x}'(t, r)]$.*

Lets suppose that $L_p^{\mathcal{F}^1}(t_0, T)$, $1 \le p \le \infty$ stands for the set of all fuzzy-valued measurable functions on $[t_0, T]$ and $C^{\mathcal{F}^1}([t_0, T])$ is the space of continuous functions over $[t_0, T]$. Hence, the fuzzy Caputo derivative is defined as follows: