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# On Lie symmetry analysis and invariant subspace methods of coupled time fractional partial differential equations

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ABSTRACT

Lie symmetry analysis and invariant subspace methods of differential equations play an important role separately in the study of fractional partial differential equations. The former method helps to derive point symmetries, symmetry algebra and admissible exact solution, while the later one determines admissible invariant subspace as well as to derive exact solution of fractional partial differential equations. In this article, a comparison between Lie symmetry analysis and invariant subspace methods is presented towards deriving exact solution of the following coupled time fractional partial differential equations: (i) system of fractional diffusion equation, (ii) system of fractional KdV type equation, (iii) system of fractional Whitham-Broer-Kaup's type equation, (iv) system of fractional Boussinesq-Burgers equation and (v) system of fractional generalized Hirota-Satsuma KdV equation.



## 1. Introduction

The subject of fractional calculus have gained considerable attention and importance during the past three decades, mainly due to its applications in numerous seemingly diverse fields of science and engineering. Also, the investigation of fractional differential equations(FDEs) have received much interest due to the exact description of nonlinear phenomena of many real-time problems. In reality, a physical phenomenon may depend not only on the time instant but also on the previous time history, and so FDEs have obtained considerable popularity and importance as generalizations of integer-order differential equations, which can be successfully modeled by using the theory of derivatives and integral of arbitrary order [1–6]. Considerable number of analytic techniques have been developed to deal with nonlinear differential equations during the past few decades. Some of the direct approaches are (i) multiple exp-function method [7] through which three wave solutions in $(3+1)$-dimensions could be derived [8]; (ii) Hirota bilinear technique which plays a significant role towards the construction of soliton solutions for nonlinear partial differential equations (PDEs) [9–11] and symbolic computations are used to generate lump solutions to many nonlinear wave equations including the Kadomtsev-Petviashvili (KP) equation [12]. However, the study of differential equations of fractional order has been handicapped due to the absence of well-defined analytic techniques to deal with them.

In recent years, both mathematicians and physicists have made many significant progress in this direction and developed some ad hoc but effective methods such as Adomian decomposition method [13], differential transform method [14], variational iteration method [15], function-expansion method of separation variables [16] and so on to deal with FDEs. However, recent investigations show that the Lie symmetry analysis and invariant subspace provide effective, powerful and systematic methods to derive exact solutions of time fractional PDEs. In the beginning of nineteenth century, Norwegian mathematician Sophus Lie was initially advocated the Lie symmetry analysis method and was further developed by Ovsianikov [17] and others [18–21]. Exploiting the Lie point symmetries one can derive group invariant solutions for differential equations. This has been demonstrated for many nonlinear PDEs including nonlinear Schrödinger (NLS) equation [22]. The Lie symmetry analysis has been extended to FDEs by Buckwar et al. [29] (see also [23–33]). The applicability of this method to FDEs has been illustrated in [17–21].

The invariant subspace method was originally introduced by Galaktionov and Svirshchevskii [34] for PDEs. The usefulness of the invariant subspace method for nonlinear PDEs has been demonstrated by many authors [35–40]. Ma [35] has shown that how one could obtain a largest possible solutions for nonlinear PDEs from the invariant subspace method [36] (see also [37–39]). Recently, this method has extended by Gazizov et al. [41] for time fractional PDEs (see also [42–47]). Here we would like to point out that

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only very few applications of coupled nonlinear system of time fractional PDEs have been investigated through the Lie symmetry analysis method. Also, to best of our knowledge, the invariant subspace method has not been extended to $m$-component coupled system of time fractional PDEs. The main objective of this work is to provide a comparison between Lie point symmetry approach and invariant subspace method towards deriving exact solution of the following coupled time fractional PDEs namely (i) system of fractional diffusion equation, (ii) system of fractional KdV type equation, (iii) system of fractional Whitham-Broer-Kaup's type equation, (iv) system of fractional Boussinesq-Burgers equation and (v) system of fractional generalized Hirota-Satsuma KdV equation.

The rest of article is systematized as follows: In Section 2, we begin with some basic definitions and properties of fractional calculus and explain how to derive the Lie point symmetries and admissible invariant subspaces for a coupled system of time fractional PDEs with Riemann-Liouville fractional derivative. Also, we provide the salient features of the invariant subspace method applicable to $m$-component coupled system of time fractional PDEs. In Section 3, we derive Lie point symmetries and admissible exact solution of the above mentioned coupled system of time fractional PDEs. In Section 4, we demonstrate that the invariant subspace helps to derive more than one exact solution of the above mentioned coupled system of time fractional PDEs if exists. In Section 5, we give a comparison of Lie symmetry analysis and invariant subspace methods and results of our investigation.

## 2. Preliminaries

In this section, we first recall some basic definitions and results of the fractional calculus. We also present a brief details of the Lie symmetry analysis and invariant subspace methods for coupled system of time fractional PDEs.

**Definition 2.1.** The Riemann-Liouville fractional differential operator of order $\alpha > 0$ of the function $\varphi(t) \in L^1[(a,b),\mathbb{R}_+]$, denoted by ${}_0D_t^\alpha$, is defined by [1]

$$\begin{aligned} {}_0D_t^\alpha \varphi(t) &= D^n \; {}_0I_t^{n-\alpha}\varphi(t) \\ &= \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\upsilon)^{(n-\alpha-1)}\varphi(\upsilon)d\upsilon, & \text{if } n-1<\alpha<n,\ n\in\mathbb{N} \\ \varphi^{(n)}(t), & \text{if } \alpha = n\in\mathbb{N} \end{cases} \end{aligned} \tag{2.1}$$

for $t > 0$.

If $\alpha = 0$, then ${}_0D_t^\alpha\varphi(t) = \varphi(t)$. For simplicity, we denote the operators ${}_0D_t^\alpha$ and ${}_0I_t^\alpha$ respectively as $D_t^\alpha$ and $I_t^\alpha$.

**Note 1.** Leibniz formula for the Riemann-Liouville fractional derivative of continuous functions $u_1(x, t)$ and $u_2(x, t)$ read

$$D_t^\alpha(u_1(x,t)u_2(x,t)) = \sum_{r=0}^{\infty}\binom{\alpha}{r}D_t^{\alpha-r}u_1(x,t)D_t^r u_2(x,t),\ \ \alpha>0, \tag{2.2}$$

where $\binom{\alpha}{r} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)\Gamma(r+1)}$.

**Definition 2.2.** The Erdélyi-Kober fractional differential operator $\left(\mathcal{P}_\delta^{\nu,\kappa}\psi\right)$ is given by [6,26]

$$\left(\mathcal{P}_\delta^{\nu,\kappa}\psi\right)(\omega) := \prod_{r=0}^{m-1}\left(\nu + r - \frac{1}{\delta}\omega\frac{d}{d\omega}\right)\left(\mathcal{K}_\delta^{\nu+\kappa,m-\kappa}\psi\right)(\omega),$$
$$\omega>0,\ \delta>0,\quad \kappa>0, \tag{2.3}$$

$m = \begin{cases} [\kappa]+1, & \text{if } \kappa\notin\mathbb{N} \\ \kappa, & \text{if } \kappa\in\mathbb{N} \end{cases}$, where

$$\left(\mathcal{K}_\delta^{\nu,\kappa}\psi\right)(\omega) := \begin{cases} \frac{1}{\Gamma(\kappa)}\int_1^\infty (u-1)^{\kappa-1}u^{-(\nu+\kappa)}\psi(\omega u^{\frac{1}{\delta}})du, & \kappa>0, \\ \psi(\omega), & \kappa = 0 \end{cases} \tag{2.4}$$

is the Erdélyi-Kober fractional integral operator.

**Note 2.** The Laplace transformation of Riemann-Liouville fractional derivative of the function $\varphi(t)$ of order $\alpha > 0$ is

$$\mathrm{L}\{D^\alpha\varphi(t)\} = s^\alpha\bar{\varphi}(s) - \sum_{r=0}^{n-1}s^r D^{\alpha-r-1}\varphi(0),$$
$$n-1<\alpha\le n,\quad n\in\mathbb{N},\quad \Re(s)>0.$$

**Definition 2.3.** A two-parameter function of Mittag-Leffler type defined by the series expansion

$$\mathbf{E}_{\beta,\gamma}(z) = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\beta r+\gamma)},\quad \beta,\gamma,z\in\mathbb{C},\quad \Re(\beta)>0,\quad \Re(\gamma)>0.$$

**Note 3.** The Laplace transformation of the function $t^{\beta m+\gamma-1}\mathbf{E}_{\beta,\gamma}^{(m)}(\pm bt^\beta)$ is

$$\mathrm{L}\left\{t^{\beta m+\gamma-1}\mathbf{E}_{\beta,\gamma}^{(m)}(\pm bt^\beta)\right\} = \frac{m!s^{\beta-\gamma}}{(s^\beta\mp b)^{m+1}},\quad \Re(s)>|b|^{\frac{1}{\beta}},$$

where $\mathbf{E}_{\beta,\gamma}^{(m)}(x) = \frac{d^m}{dx^m}\mathbf{E}_{\beta,\gamma}(x)$.

Next, we recall the convolution theorem for Laplace transformation.

**Theorem 2.4.** *If* $\mathrm{L}\{\psi(t)\} = \overline{\psi}(s)$ *and* $\mathrm{L}\{\varphi(t)\} = \overline{\varphi}(s)$, *then*

$$\mathrm{L}^{-1}\{\overline{\psi}(s)\overline{\varphi}(s)\} = \psi(t)\star\varphi(t),$$

*where* $\psi(t)\star\varphi(t)$ *is called the convolution of* $\psi(t)$ *and* $\varphi(t)$ *and is defined by the integral*

$$\psi(t)\star\varphi(t) = \int_0^t\psi(t-r)\varphi(r)dr = \int_0^t\varphi(t-r)\psi(r)dr.$$

### 2.1. Lie symmetry analysis for coupled time fractional PDEs

We present below a brief details of Lie symmetry analysis for coupled time fractional PDEs with two independent variables. Consider a two-coupled time fractional PDEs having the following form

$$\begin{aligned} \frac{\partial^\alpha u_1}{\partial t^\alpha} &= G_1\left(x,u_1,u_2,u_1^{(1)},u_2^{(1)},\ldots,u_1^{(k_1)},u_2^{(k_2)}\right), \\ \frac{\partial^\alpha u_2}{\partial t^\alpha} &= G_2\left(x,u_1,u_2,u_1^{(1)},u_2^{(1)},\ldots,u_1^{(k_1)},u_2^{(k_2)}\right),\quad \alpha>0, \end{aligned} \tag{2.5}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}(.)$ is a fractional time derivative in the Riemann-Liouville sense. In the remaining part of the article, we use the following notations:

$$u_q = u_q(x,t),\quad u_q^{(j)} = \frac{\partial^j u_q(x,t)}{\partial x^j},$$
$$j=1,2,\ldots,k_q,\quad q=1,2,\ldots,m;$$