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Bifurcations of spatiotemporal structures in a medium of FitzHugh–Nagumo neurons with diffusive coupling

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ABSTRACT

We study the boundaries of existence of traveling waves and stationary spatial structures in an active medium model by varying the control parameters. The medium is represented by a ring of diffusively coupled FitzHugh–Nagumo neurons, which, when uncoupled, can demonstrate excitable, self-sustained oscillatory or bistable dynamics depending on control parameter values. The dynamical regimes realized in the medium are compared with those ones observed in an individual FitzHugh–Nagumo neuron. Possible bifurcations of traveling waves are analyzed when the dynamics of the medium elements changes. We also explore the influence of the relaxation level of FitzHugh–Nagumo neurons on the medium dynamics.

1. Introduction

Studying the dynamics of nonlinear media and distributed systems is one of the current research trends in nonlinear dynamics. A number of monographs [1–3] are devoted to the problems of complex spatial structure formation. A so-called active medium, in which self-sustained wave processes are observed, is of particular interest for researchers [1]. Active media models and their discrete analogs in the form of lattices of interacting active elements are widely used to solve various problems in biophysics and neurodynamics [4–6], chemistry [7,8], epidemiology [9], and other scientific fields.

Spatial structure formation is a manifestation of the synergetics, which is related to synchronization of a group of interacting partial elements of a distributed system. The features of interacting elements together with the type of their interaction play an important role in the complex structure formation. Recently, a new type of spatial structures, which is observed in ensembles with nonlocal coupling and called chimera state [10,11], has become an object of intensive researches [12–17]. The chimera state consists of domains of spatially coherent and incoherent dynamics and, apparently, can be treated as a specific type of cluster synchronization. Recently there were works devoted to chimeras in ensembles with a more complex structure of nonlocal couplings [18,19], as well as in multilayer networks with a different topology [20].

In many cases a continuous medium can be described by a distributed system with a large number of locally coupled small-size elements. In this case, three types of active media can be distinguished, i.e., self-sustained oscillatory, excitable and bistable, and thus, a medium element can be represented by a self-sustained oscillator, an excitable system or a bistable oscillator with two stable states, respectively. All these media can demonstrate self-sustained wave phenomena [1,2], which can have essential differences. The elements of a self-sustained medium always exhibit stationary oscillations in a relevant regime. Certain specific conditions are needed to produce similar oscillations in an excitable medium. These conditions must provide the return of an excitation pulse to an element after some relaxation time. So, stationary excitation waves travel in a ring of excitable elements [4,21–25]. Wave structures in the form of spiral waves can be observed in two- and three-dimensional excitable lattices for specific initial conditions [1,4,26–28].

Traveling waves are not always realized in distributed systems of bistable elements. For example, wave regimes cannot be exhibited by a ring of Duffing oscillators with diffusive coupling, and only the unidirectional interaction can cause the occurrence of traveling waves in this system [29,30]. On the contrary, a variety of wave modes with a different wavelength can be observed in a medium model of diffusively coupled FitzHugh–Nagumo neurons in the bistable regime (see [31,32]).

The FitzHugh–Nagumo neuron is known to be the simplest neuron model and a classical example of an excitable system [33,34]. At the same time, for a specific form of the system equations, either bistable regime or self-sustained oscillations can also be realized in the FitzHugh–Nagumo neuron depending on its control

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parameters. Thus, it is important to analyze the dynamical behavior of a medium of FitzHugh–Nagumo neurons and its changes when the parameters of the medium element are varied. Recently, the specific common features and differences of traveling waves have been found in self-sustained oscillatory, excitable and bistable regimes [32,35]. However, they have not been sufficiently explored when the control parameters of an individual element are varied. The main objective of the present work is to carry out the study in detail and to fully complete this analysis.

2. System under study

The FitzHugh–Nagumo (FHN) neuron is the basic model for our study. It is an approximate two-dimensional neuron model which was obtained by simplifying the Hodgkin-Huxley model [36]. The FHN neuron being one of the simplest model of an excitable system is widely used to simulate neuronal ensembles and fibers in biophysics [4,5,37]. Different forms of the FHN neuron equations are usually used. We consider the following system of equations:

$$\varepsilon\dot{x} = x - \alpha x^3 - y,\\ \dot{y} = \gamma x - y + \beta. \tag{1}$$

Here x and y are the dimensionless variables. The model (1) describes the regenerative self-excitation of a voltage on a cell membrane (the x variable) as a result of nonlinear positive feedback and also the regeneration due to a linear negative current feedback (the y variable). The dimensionless parameters $\alpha, \beta, \gamma, \varepsilon$ control the neuron dynamics, and ε is usually small. The system (1) can operate in three regimes, namely, self-sustained oscillatory, excitable and bistable, depending on its parameter values.

An ensemble of locally interacting neurons (1) can serve as a sufficiently universal model of an active medium which can be self-sustained oscillatory, excitable or bistable in accordance with parameter values of its elements. In our paper we study a ring of FHN neurons with diffusive coupling, which is described by the following system of equations:

$$\varepsilon\dot{x}_j = x_j - \alpha x_j^3 - y_j + k(x_{j-1} + x_{j+1} - 2x_j),\\ \dot{y}_j = \gamma x_j - y_j + \beta,\\ x_{j+N}(t) = x_j(t),\; y_{j+N}(t) = y_j(t)\; j = 1, \ldots, N. \tag{2}$$

Here N is the number of the ring elements, and k is the coupling strength. The discrete medium model (2) can be transformed to a model with a continuous spatial coordinate s by applying the following procedure. If the system length in space is L, then the medium element has the size Δs (the discretization step on the spatial coordinate). In the limits $N\to\infty$ and $\Delta s\to 0$, a continuous medium with the diffusive coefficient $\sigma = k\cdot(\Delta s)^2$ can be represented as follows:

$$\varepsilon\dot{x} = x - \alpha x^3 - y + \sigma\frac{\partial^2 x}{\partial s^2},\\ \dot{y} = \gamma x - y + \beta,\\ x(s \pm L, t) = x(s, t),\; y(s \pm L, t) = y(s, t),\; i = 1, \ldots, N. \tag{3}$$

The basic features of the medium 3) behavior in different regimes have been studied in [32,35]. However, the bifurcation analysis has not been carried out and the medium behavior has not been compared with the dynamics of a single FHN neuron.

3. Bifurcation analysis of dynamical regimes in a single FHN neuron

We start by considering various regimes in the single FHN neuron (1) for the fixed values $\alpha = 1/3$, $\varepsilon = 0.2$ and when β and γ are varied. The bifurcation diagram for (1) is shown in Fig. 1 in the (β, γ) parameter plane.

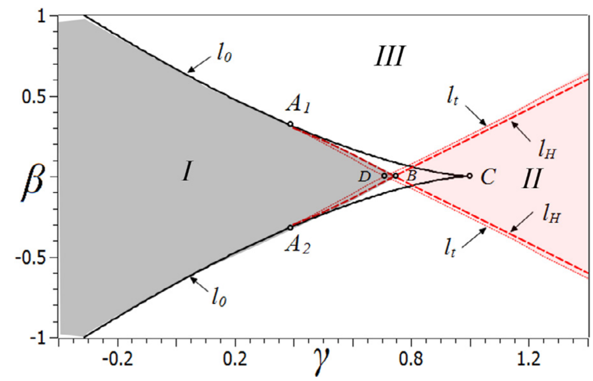

Fig. 1. Bifurcation diagram for the system (1) in the (β, γ) parameter plane for $\alpha = 1/3$ and $\varepsilon = 0.2$. *I* is the region with two stable equilibrium points; *II* is the region of self-sustained oscillations; *III* is the region with a single stable equilibrium point; l_0 is the line of the fold bifurcation of equilibria; l_{AH} is the line of the subcritical Andronov–Hopf bifurcation; l_t is the line of the fold bifurcation for limit cycles; l_h is the line of the homoclinic bifurcation.

Three different regions are highlighted in the bifurcation diagram (Fig. 1: the bistable region with two stable equilibrium points (*I*), the self-sustained oscillation region (*II*) and the region with a single stable equilibrium point (*III*). The excitable regime is observed near the self-sustained oscillation generation threshold. One of its features consists in the emergence of activity spikes when the external force exceeds a certain threshold. Either one or three equilibrium points exist in the system (1) depending on β and γ values. The solid lines in the diagram Fig. 1 correspond to a fold bifurcation of the equilibrium points. The relevant bifurcational values can be estimated analytically as follows:

$$\beta = \pm\frac{2}{3}\sqrt{\frac{1-\gamma}{3\alpha}}(1-\gamma). \tag{4}$$

Point C denotes the casp point. For $\alpha = 1/3$ we have

$$\beta = \pm\frac{2}{3}\sqrt{(1-\gamma)}(1-\gamma). \tag{5}$$

The location of the bifurcation lines in Fig. 1 is independent on the parameter ε.

One of the equilibrium points involved in the fold bifurcation is always a saddle and the other point can be either stable or unstable. The fold bifurcation lines are related to a saddle-node bifurcation to the left of points A_1,A_2 and to a saddle-repeller one to the right of them. When β is small, then the FHN neuron exhibits the bistable dynamics with two stable equilibrium points. In a certain range of γ values and when increasing $|\beta|$, each of the stable points loses its stability as a result of a subcritical Andronov–Hopf bifurcation (the dashed lines in Fig. 1). Thus, bistability region *I* are bounded by the lines of the fold bifurcation (to the left of points A_1, A_2) and of the subcritical Andronov–Hopf bifurcation (in the direction from A_1, A_2 to B).

The Andronov–Hopf bifurcation in the system (1) (the dotted lines in Fig. 1) is subcritical for all the considered parameter values and is related to the contraction of an unstable cycle to the equilibrium point. The dotted lines to the right of point D correspond to the fold bifurcation of limit cycles. A pair of limit cycles (stable and unstable) emerges when crossing these lines from left to right. Thus, these lines can determine the boundaries of self-sustained oscillation region *II*. The dotted lines to the left of point D indicate a homoclinic bifurcation. Only one unstable limit cycle is born in this case. Since the lines of fold and homoclinic bifurcations are located very closely to the Andronov–Hopf bifurcation lines, the evolution of cycles is difficult to detect.