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Lyapunov analysis of the spatially discrete-continuous system dynamics

Vladimir A. Maximenko[a], Alexander E. Hramov[a,b], Alexey A. Koronovskii[b,*], Vladimir V. Makarov[a,b], Dmitry E. Postnov[b], Alexander G. Balanov[c]

[a] *Yurij Gagarin State Technical University of Saratov, Politehnicheskaya, 77, Saratov 410054, Russia*
[b] *Saratov State University, Astrakhanskaya, 83, Saratov 410012, Russia*
[c] *Department of Physics, Loughborough University, Loughborough LE11 3TU, UK*

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ABSTRACT

The spatially discrete-continuous dynamical systems, that are composed of a spatially extended medium coupled with a set of lumped elements, are frequently met in different fields, ranging from electronics to multicellular structures in living systems. Due to the natural heterogeneity of such systems, the calculation of Lyapunov exponents for them appears to be a challenging task, since the conventional techniques in this case often become unreliable and inaccurate. The paper suggests an effective approach to calculate Lyapunov exponents for discrete-continuous dynamical systems, which we test in stability analysis of two representative models from different fields. Namely, we consider a mathematical model of a 1D transferred electron device coupled with a lumped resonant circuit, and a phenomenological neuronal model of spreading depolarization, which involves 2D diffusive medium. We demonstrate that the method proposed is able reliably recognize regular, chaotic and hyperchaotic dynamics in the systems under study.

1. Introduction

It is quite common in complexity science, when a spatially extended media with infinitely many degrees of freedom interacts with a dynamical system localized in space and having a finite number of degrees of freedom. The mathematical models of such discrete-continuous systems (DCS) are composed of partial differential equations (PDEs) coupled with ordinary differential equations (ODEs).

The models that fall to the class of discrete-continuous systems arise in many applications from different research fields ranging from life sciences to information processing and electronics. Incomplete list of such problems includes modeling of drug delivery to biological tissues [1], neural dynamics [2], mitochondrial swelling [3], intracellular signaling [4], cortical spreading depression [5], quantum information processing [6], active semiconductor media interacting with discrete elements [7], lumped circuits coupled to a transmission line [8], multiscale continuum mechanics [9].

The similar class of model systems appears in the number of biophysical problems, where the hemodynamics, which is often described by Navier–Stokes PDEs, is considered together with the time-variable system-wide quantities, e.g., the blood pressure or electrocardiography (ECG) [10,11].

Due to the importance of the spatially discrete-continuous models for the different research fields, the specialized solution algorithms were developed (e.g. [12]). However, there is a clear shortfall of the tools available for stability analysis of such dynamical systems. DCSs are often analysed with the help of methods, developed for systems with finite number of the degrees of freedom. In this context the original spatially-distributed subsystem can be described by the set of ODEs based on lattice model [13] or Laplace transform method [14]. In electronics the dynamics of DCSs are often analysed by the consideration of the subsystems with finite number of the degrees of freedom and spatially-extended subsystem separately [15,16]. These approaches, obviously, have their specific limitations. In particular, transition to the lattice model can potentially affects the system dynamics in an unpredictable way [13], while the consideration of the dynamical regimes taking place in finite-dimensional subsystem may not reflect the key features of spatiotemporal behaviour of spatially extended subsystem [17].

* Corresponding author.
E-mail address: alexey.koronovskii@gmail.com (V.A. Maximenko).

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The most promising approach for the stability analysis of DCSs is based on the calculation of Lyapunov exponents (LEs). The use of such tool makes the significant progress in study of the finite-dimensional flow systems [18,19], discrete maps [20] and time-series [21] (including the cases with the presence of noise, see, e.g, [22]). In recent works Lyapunov exponents are applied for analysis of non Hermitian Hamiltonian systems [23] and neural systems [24]. In the case of spatiotemporal dynamics the calculation of LEs is more complicated [25,26]. At the same time, the recent results on Lyapunov analysis of the extended media, described within the framework of hydrodynamic approximation has shown a great potential of this technique for the quantitative assessment of chaotic behavior [26,27], detection of hyper-chaotic regimes [28] and identification of the synchronous modes in coupled spatially extended elements [28] as well as networks of interacting spatially extended units [29,30].

It should be noted, the existent methods of the LEs calculation either for the finite-dimensional systems [31] (such as flows or maps) or spatially extended media [28] cannot be directly applied to DCSs, i.e., to the systems consisting of both the spatially extended and concentrated in space subsystems. The main problem here is that the reference states of such systems are determined simultaneously by two significantly different types of variables, namely, by the variables depending only on time (which correspond to the finite dimensional subsystems) and by the functions which depend both on time and space coordinates (they represent the spatially extended subsystems). This makes impossible the straightforward implementation of the normalization and orthogonalization procedures, developed for the finite dimensional [31] and spatially-extended [28] systems and, as the results, the accurate estimation of Lyapunov exponents.

In the present paper we introduce an approach allowing to calculate the spectrum of LEs for discrete-continuous dynamical systems. In order to illustrate the universality and capability of the proposed method as well as its relevance, we apply the developed approach to analyze the stability of dynamical regimes in two radically different exemplary DCS that came from different research fields.

First, we perform the Lyapunov stability analysis of the charge dynamics in a finite-dimensional dynamical circuit, where a spatially extended 1D media is included as an nonlinear element [7]. The latter is described by a set of the coupled Poisson and continuity equations, whereas the circuit is described with the help of non-stationary Kirchhoff equations.

Next, we consider an example from different research area. Namely, we analyze the dynamics of a phenomenological model of spreading depolarization [5], that is composed of a set of FitzHugh–Nagumo (FHN) oscillators (model neurons) coupled through 2D diffusive media that describe the extracellular spreading of depolarizing substances.

In both cases the Lyapunov analysis allowed us to reveal and quantify the transitions between the regular and chaotic dynamics with variation of the control parameters.

The paper has the following structure. The approach to calculation of the spectrum of LEs for DCS is described in Section 2. The dynamics of the RLC-circuit connected with the semiconductor transferred electron device (TED) is described and analyzed in Section 3. Section 4 is devoted to the Lyapunov stability analysis of the model of the spreading depolarization. The final remarks and conclusions are given in Section 5.

2. Calculation of the Lyapunov exponents for spatially discrete-continuous systems

Let us consider an arbitrary DCS, which is described by a set of coupled PDEs and ODEs. The state of the spatially extended medium modeled by PDEs is supposed to be defined by N variables, each being a function of both the displacement vector $\mathbf{r}$ and time t

$$\Phi_1(\mathbf{r},t),\,\Phi_2(\mathbf{r},t),\ldots,\,\Phi_{N-1}(\mathbf{r},t),\,\Phi_N(\mathbf{r},t),\quad \mathbf{r}\in\mathbb{R}^D,\quad 0\le t\le\infty, \tag{1}$$

D is the dimension of the space (in our study $D=1$ for the system considered in Section 3 and $D=2$ for the discrete-continuous model of the spreading depression discussed in Section 4). The variables depending only on time

$$\Theta_1(t),\,\Theta_2(t),\ldots,\,\Theta_{M-1}(t),\,\Theta_M(t),\quad 0\le t\le\infty. \tag{2}$$

describe the state of the subsystems with $M/2$ degrees of freedom defined by ODEs.

In order to characterize the stability of the DSC dynamics, one has to trace the evolutions of the system state (in our case it is $\mathbf{U}(\mathbf{r},t)=(\Phi_1(\mathbf{r},t),\ldots,\Phi_N(\mathbf{r},t),\Theta_1(t),\ldots\Theta_M(t))^T$) and analyse how a linear perturbation of this state changes with time. However, this procedure for the case when state variables depend only on time [31,32] is significantly different from the case, when the state variables depend both on time and displacement [27,28]. In our situation we deal with a mix of two type of the variables mentioned above, which prevents a direct application of the convention routines. To overcome this conceptual obstacle, we propose to consider the variables (2) as the spatially extended ones, i.e.,

$$\Psi_k(\mathbf{r},t)=\Theta_k(t),\quad k=\overline{1,M}. \tag{3}$$

In this case the state of the spatially discrete-continuous system may be considered as

$$\mathbf{U}(\mathbf{r},t)=(\Phi_1(\mathbf{r},t),\ldots,\Phi_N(\mathbf{r},t),\,\Psi_1(\mathbf{r},t),\ldots,\Psi_M(\mathbf{r},t))^T, \tag{4}$$

and the evolution operator

$$\hat{L}(\mathbf{U}(\mathbf{r},t)), \tag{5}$$

determines the spatiotemporal behavior of the system state. This evolution operator consists typically of coupled ordinary differential equations and partial differential equations determining the evolution of localized in space subsystems and spatially extended media, respectively. E.g., for the RLC-TED circuit considered in Section 3 the evolution operator (5) consists of ODEs (19) and PDEs (20)–(21) with the boundary conditions (24). Assume that $\mathbf{r}=x$ in the case of $D=1$, $\mathbf{r}=(x,y)$ when $D=2$ and $\mathbf{r}=(x,y,z)$ for $D=3$.

The numerical algorithms for the LE calculation are usually based on the analysis of the perturbation $\mathbf{V}(\mathbf{r},\,t)$ of the reference state $\mathbf{U}(\mathbf{r},\,t)$ and the calculation of the increment/decay rate. To estimate the K largest Lyapunov exponents Λ_i, $i=1,\ldots,K$, one has to consider a set of orthogonal perturbations $\mathbf{V}_i(\mathbf{r},t), i=1,\ldots,K$. In this case, the Lyapunov exponents characterize the exponential growth/decay of K orthogonal modes of $\mathbf{U}(\mathbf{r},\,t)$. Each perturbation $\mathbf{V}_i(\mathbf{r},\,t)$ is defined as

$$\mathbf{V}_i(\mathbf{r},t)=(\tilde{\phi}_1^i(\mathbf{r},t),\ldots,\tilde{\phi}_N^i(\mathbf{r},t),\,\tilde{\psi}_1^i(\mathbf{r},t),\ldots,\tilde{\psi}_M^i(\mathbf{r},t))^T,\quad i=\overline{1,K} \tag{6}$$

assuming that all $\tilde{\psi}_k^i(\mathbf{r},t)$ depend only on time, i.e.,

$$\tilde{\psi}_k^i(\mathbf{r},t)\equiv\tilde{\theta}_k^i(t),\quad \forall k,\ \forall i. \tag{7}$$

The perturbations introduced must initially be orthogonal and normalized. The orthogonality condition reads

$$(\mathbf{V}_i(\mathbf{r},0),\mathbf{V}_j(\mathbf{r},0))=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases} \tag{8}$$

where the brackets $(\cdot,\cdot)$ denote the scalar product