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Solvability of chaotic fractional systems with 3D four-scroll attractors

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ABSTRACT

One of the questions that has recently predominated the literature is the generation and modulation of strange chaotic attractors, namely the ones with multi scrolls. The fractional theory might be useful in addressing the questions. We use the Caputo fractional derivative together with Haar wavelet numerical scheme to investigate a three-dimensional system that generates chaotic four-wing attractors. Some conditions of stability at the origin (the trivial equilibrium point) are provided for the model. The error analysis shows that the method converges and is concluded thanks to Fubini–Tonelli theorem for non-negative functions and the Mean value theorem for definite integrals. Graphical simulations, performed for some different value of the derivative order α show existence, as expected, of chaotic dynamics characterized by orbits with four scrolls, typical to strange attractors. Hence, fractional calculus appears to be useful in generating and modulating chaotic multi-wing attractors.

1. Introduction to the model

Even though a huge interest for fractional differentiations and their properties has only resurfaced during the last two decades, fractional calculus remains a scientific domain as old integer order calculus is. Many authors have applied it in various processes related to real life phenomena, such as acoustic dissipation, viscoelastic systems, mathematical epidemiology, continuous time random walk, biomedical engineering, porous media, control theory, Levy statistics, fractional Brownian, dielectric polarization, fractional signal and image processing, electrolyteelectrolyte polarization, fractional filters motion and nonlocal phenomena [1–9]. Most of the models used in those analysis are non linear and require sophisticated techniques to solve them. Hence, number of numerical methods for the solution of fractional differential equations have been developed and proposed in numerous works, in order to provide an improved description of the phenomenon under investigation. Common numerical methods include finite difference method, variational iteration method, Crank–Nicholson method, adomian or homotopy analysis and lastly the one of our interest in this paper: wavelet method [10–18]. Wavelet analysis appears to be relatively new in mathematical analysis theory but is catching interest among scientist, especially those specialized in fluid flow, applied in signal and image manipulation and numerical analysis, etc.

On the other side, the scientific academy has seen, during the years, the development and simulation of the so called strange attractors whose unique particularity is to exhibit attractor with a fractal structure [19–21]. Edward Lorenz [22] is one of the first to propose strange attractor, Lorenz attractor. However, there are number of other systems of equations that generate strange attractors leading to chaotic dynamics. Few examples include the Rössler attractor [23] and Hénon attractor [24], Arneodo Attractor [25], Lu-chen attactor [26], etc, and lastly the one of our interest in this paper: Four-wing attactor. This paper aims to assess the effect resulted from a combination of fractional derivative and those strange systems of equations. Whence, the whole analysis conducted here consists of exploring the existence of four-wing attractor and stability results for the model (1.3) here below, that belongs to the same family as the chaotic Rössler system [23,27,28] given as

$$\begin{cases} \dfrac{d}{dt}x(t) = -y - z, \\ \dfrac{d}{dt}y(t) = x + ay, \\ \dfrac{d}{dt}z(t) = bx + z(x - c), \end{cases} \tag{1.1}$$

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or the Lorenz system [22,27,28]

$$\begin{cases} \dfrac{d}{dt}x(t) = \sigma(y - x), \\ \dfrac{d}{dt}y(t) = x(\rho - z) - y, \\ \dfrac{d}{dt}z(t) = xy - \delta z, \end{cases} \tag{1.2}$$

with $\alpha \in [0;1]$, $\beta \in (0, +\infty)$, $t > 0$ where $x = x(t)$, $y = y(t)$, $z = z(t)$ represent the system state and σ, ρ, δ are real constants parameterizing the system.

The model of our interest reads as

$$\begin{cases} D_t^\alpha x(t) = ax + cyz, \\ D_t^\alpha y(t) = bx + dy - xz, \\ D_t^\alpha z(t) = ez + fxy, \end{cases} \tag{1.3}$$

where $a, b, d, e \in \mathbb{R}$, $c > 0$ and $f < 0$ with $cf \neq 0$. $x = x(t)$, $y = y(t)$, $z = z(t)$ represent the system state and a_1, a_2, b_1, c_1 are real constants parameterizing the system. The term D_t^α represents a fractional derivative. In the next section, a comprehensive definition of the fractional derivative we employ, namely the Caputo derivative and more other details with properties are provided. Our approach is to fully analyze the model (1.3) for any order $\alpha \in [0;1]$. More precisely, we solve the model using the numerical method of Haar wavelets that is described in Section 3 below. The goal is to refute or not the (non)existence of a chaotic four-wing attractor for (1.3). Before that, let us recall the following

Theorem 1.1. *It is impossible for the system (1.3) to generate a chaotic four-wing attractor when $\alpha = 1$ and $b = 0$.*

Proof. The proof follows from [28, Theorem 1] and the fact that

$$D_t^1 u(t) \sim \frac{du(t)}{dt}. \tag{1.4}$$

□

Hence for $\alpha = 1$, the model (1.3) reduces to the system

$$\begin{cases} x'(t) = ax + cyz, \\ y'(t) = bx + dy - xz, \\ z'(t) = ez + fxy \end{cases} \tag{1.5}$$

System (4.2) was introduced in [28,29] proved to be chaotic in the same level as Lorenz or Rössler equations are. Moreover, it generates a four-wing chaotic attractor with less terms in the system equations compared to other models. Then, let us analyze the extended model (1.3) and exhibit the shape of the solutions in order to compare with those of (4.2).

2. A note on derivative with non-integer order [11,27,30–33]

In this particular domain of calculus, the most popular definitions of derivatives with non-integer order remain the Riemann–Liouville derivative (RLFD) and Caputo derivative. The first was named after the work of Bernhard Riemann and Joseph Liouville more than a century and a half ago. Their main idea started with the following integral of order α

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{2.1}$$

based on Euler transform when applied to analytic function and Cauchy's formula for calculating iterated integrals. Hence, the RLFD of order α was defined for any $t > 0$ as

$$\mathcal{D}_t^\alpha f(t) = \frac{d^n}{dt^n} I^{n-\alpha} f(t), \quad n - 1 < \alpha \leq n \tag{2.2}$$

where $n \in \mathbb{N}$, $-\infty \leq a < t$, $b > a$ and $f : (a, b) \longrightarrow \mathbb{R}$ an arbitrary real and locally integrable function. After that, in 1967, Michele Caputo proposed another definition closely related to the previous one and given (for $n = 1$) as

$$D_t^\alpha f(t) = I^{1-\alpha} \frac{d}{dt} f(t), \quad 0 < \alpha \leq 1 \tag{2.3}$$

where the unknowns are the same as in (2.2), except the function f that is from the first order Sobolev space

$$H^1(a, b) = \left\{ f : f, \frac{d}{dt} f \in L^2(a, b) \right\}. \tag{2.4}$$

Recently, more investigations conducted by Caputo and Fabrizio [31] pointed out another definition, the Caputo–Fabrizio fractional derivative given by

$${}^{cf}D_t^\alpha f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t \dot{f}(\tau) \exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau, \tag{2.5}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. Soon after that Losada and Nieto [32] improved this definition as

$${}^{cf}D_t^\alpha f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \dot{f}(\tau) \exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau. \tag{2.6}$$

and defined a more suitable fractional integral that reads as:

$${}^{cf}I_t^\alpha f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(\tau) d\tau, \tag{2.7}$$

$\alpha \in [0, 1]$ $t \geq 0$. This anti-derivative represents sort of average between the function f and its integral of order one. In the same momentum, Goufo and Atangana [4,11] propose the New Riemann–Liouville fractional order derivative given for $\alpha \in [0, 1]$ by

$${}_a\mathfrak{D}_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t f(\tau) \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) d\tau \tag{2.8}$$

Again, the NRLFD is without any singularity at $t = \tau$ in comparison to the classical Riemann–Liouville fractional order derivative (2.2) and also it verifies

$$\lim_{\alpha \to 1} {}_a\mathfrak{D}_t^\alpha f(t) = \dot{f}(t) \tag{2.9}$$

and

$$\lim_{\alpha \to 0} {}_a\mathfrak{D}_t^\alpha f(t) = f(t). \tag{2.10}$$

In order to address the issue of locality that exists in the above definitions of fractional derivatives, nonlocal definitions were proposed and generalized [27,33] as follows: Let f be a function in $H^1(a; b)$; $b > a$; $\alpha \in [0; 1]$, $\beta \in (0, +\infty)$ then, the Caputo-sense one-parameter and nonlocal fractional derivative of order α is given by:

$${}^{ab}D_t^\alpha f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t \dot{f}(\tau) E_\alpha\left[-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right] d\tau =_a^{abc} D_t^\alpha f(t). \tag{2.11}$$

where $M(\alpha)$ is the same normalization function defined in (2.5) and E_α the one-parameter Mittag–Leffler function.

The Caputo-sense two-parameter and nonlocal fractional derivative of order α knowing β as a parameter is given by:

$${}^{gc}D_t^{\alpha,\beta} f(t) = \frac{\beta W(\alpha,\beta)}{(\beta-\alpha)} \int_a^t \dot{f}(\tau) (t-\tau)^{\beta-1} E_{\alpha,\beta}\left[-\frac{\alpha\beta(t-\tau)^\alpha}{\beta-\alpha}\right] d\tau, \tag{2.12}$$

where $W(\alpha, \beta)$ is a two-variable normalization function such that $W(0, 1) = W(1, 1) = 1$, and and $E_{\alpha,\beta}$ the two-parameter Mittag-Leffler function.