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Quantum white noise Gaussian kernel operators

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ABSTRACT

The quantum white noise (QWN) Gaussian kernel operators (with operators parameters) acting on nuclear algebra of white noise operators is introduced by means of QWN- symbol calculus. The quantum-classical correspondence is studied. An integral representation in terms of the QWN- derivatives and their adjoints is obtained. Under some conditions on the operators parameters, we show that the composition of the QWN-Gaussian kernel operators is a QWN-Gaussian kernel operator with other parameters. Finally, we characterize the QWN-Gaussian kernel operators using important connection with the QWN- derivatives and their adjoints.

1. Introduction

Guassian operators are important in the representation theory of infinite dimensional groups. Such operators appear in analysis and mathematical physics. As important examples of Gaussian operators, namely: the Fourier transform, the Poisson formula and the Mehler formula. Gaussian operator is by definition [13] an integral operator of the form

$$Bf(x) = \int_{y\in\mathbb{R}^n} \chi(x,y) f(y),$$

where $\chi(x, y)$ is a Gaussian distribution on $\mathbb{R}^m \times \mathbb{R}^n$ and f is a function on $\mathbb{R}^n$. It is obvious that B is a well defined operator from $S(\mathbb{R}^n)$ into $S'(\mathbb{R}^n)$. Following Neretin (see [13]) the Gaussian operators in spaces of functions of infinite numbers of variables are more important in probability and mathematical physics than finite dimension Gaussian operators. Trying to overcome this difficulty in extending the finite dimensional Gaussian operators to those in infinite dimensions, Luo and Yan (see [12]) studied such operators on infinite dimensional white noise functional spaces. The white noise analysis has been developed to an infinite dimensional distribution theory on Gaussian space (E^*, μ) as an infinite dimensional analogue of Schwartz distribution theory on Euclidean space $\mathbb{R}$ with Lebesgue measure:

$$E := \mathcal{S}(\mathbb{R}) \subset H := L^2(\mathbb{R}, dx) \subset \mathcal{S}'(\mathbb{R}) =: E^*. \tag{1}$$

Our mathematical framework of white noise analysis is the Gel'fand triple of test function space $\mathcal{F}_\theta(E^*_{\mathbb{C}})$ and generalized function space $\mathcal{F}^*_\theta(E^*_{\mathbb{C}})$:

$$\mathcal{F}_\theta(E^*_{\mathbb{C}}) \subset L^2(E^*, \mu) \subset \mathcal{F}^*_\theta(E^*_{\mathbb{C}}). \tag{2}$$

There is a formal analogy between white noise calculus and the calculus on Euclidean space based on this Gel'fand triple, e.g., rotation groups [7], Laplacians [6].

In white noise analysis, the set $\{x(t);\ t\in\mathbb{R}\}$ is taken as a coordinate system of (E^*, μ) and $\{a_t,\ a_t^*;\ t\in\mathbb{R}\}$ (annihilation and creation) is the coordinate system for white noise differential operators as homologue of the Euclidean differential basis $\{\frac{\partial}{\partial x_k};\ 1\le k\le d\}$. It is a fundamental fact that every white noise operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta(E^*_{\mathbb{C}}), \mathcal{F}^*_\theta(E^*_{\mathbb{C}}))$ admits a Fock expansion as an infinite series:

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{3}$$

where the integral kernel operator $\Xi_{l,\,m}(\kappa_{l,\,m})$ is given by

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{\mathbb{R}^{l+m}} \kappa_{l,m}(s_1,\cdots,s_l,t_1,\cdots,t_m) a^*_{s_1}\cdots a^*_{s_l} a_{t_1} \cdots a_{t_m} ds_1\cdots ds_l dt_1\cdots dt_m. \tag{4}$$

The white noise operator Ξ can be regarded as a "function" of the variables $\{a_s,\ a_t^*;\ s, t\in\mathbb{R}\}$. This intuitive idea allows to introduce the so-called quantum white noise derivatives (see Ref. [9])

$$D_t^+\Xi = \frac{\partial\Xi}{\partial a_t} \equiv [a_t, \Xi], \qquad D_t^-\Xi = \frac{\partial\Xi}{\partial a_t^*} \equiv -[a_t^*, \Xi]$$

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acting on a suitable subset of the nuclear algebra $\mathcal{L}(\mathcal{F}_\theta(E^*_{\mathbb{C}}), \mathcal{F}^*_\theta(E^*_{\mathbb{C}}))$. The set

$$\left\{D^+_s,\ D^-_t,\ (D^+_u)^*,\ (D^-_v)^*;\ s,\ t,\ u,\ v \in \mathbb{R}\right\}$$

will be taken as a quantum white noise coordinate system.

Based on an adequate definition of a QWN- symbol and an infinite dimension nuclear spaces, the main purpose of this paper is to introduce and characterize the Gaussian operators (which will called QWN-Gaussian kernel operators) acting on white noise operators using the quantum white noise coordinate system. Many important operators, such as the quantum second quantization, the quantum Fourier–Mehler transform (see [2,3,8]), etc. are QWN-Gaussian kernel operators.

The paper is organized as follows. In Section 2, we summarize the common notations, concepts and basic topological isomorphisms used throughout the paper. In Section 3, we introduce the QWN-Gaussian kernel operator. Then, we study the quantum-classical correspondance and we give important integral representation of QWN-Gaussian kernel operator. In Section 4, we show that the composition of the QWN-Gaussian kernel operators is a QWN-Gaussian kernel operator with others parameters. In Section 5, we characterize the QWN-Gaussian kernel operators using a connection with QWN- derivatives and their adjoints.

2. Backgrounds

In this section we summarize the common notations and concepts used throughout the paper which can be found in Refs. [1–5,11,14–19].

2.1. Basic Gel'fand triples

Let H be the real Hilbert space $L^2(\mathbb{R})$ of square integrable functions on $\mathbb{R}$ with norm $|\cdot|_0$. The Gel'fand triple (1) can be reconstructed in a standard way (see Ref. [14]) by the harmonic oscillator $A = 1 + t^2 - d^2/dt^2$ and H. The eigenvalues of A are $2n+2$, $n = 0, 1, 2, \cdots$, the corresponding eigenfunctions $\{e_n; n \geq 0\}$ form an orthonormal basis for $L^2(\mathbb{R})$. In fact (e_n) are the Hermite functions and therefore each e_n is an element of E. The space E is a nuclear space equipped with the Hilbertian norms

$$|\xi|_p = |A^p\xi|_0, \qquad \xi \in E, \quad p \in \mathbb{R}$$

and we have

$$E = \operatorname{projlim}_{p\to\infty} E_p, \qquad E^* = \operatorname{indlim}_{p\to\infty} E_{-p},$$

where, for $p \geq 0$, E_p is the completion of E with respect to the norm $|\cdot|_p$ and E_{-p} is the topological dual space of E_p. We denote by $N = E + iE$ and $N_p = E_p + iE_p$, $p \in \mathbb{Z}$, the complexifications of E and E_p, respectively.

Throughout the paper, we fix a Young function θ satisfies the following condition

$$\limsup_{x\to\infty} \frac{\theta(x)}{x^2} < \infty. \tag{5}$$

The polar function θ^* of θ, defined by $\theta^*(x) = \sup_{t\geq 0}(tx - \theta(t))$, $x \geq 0$, is also a Young function. For more details, see Refs. [5] and [15]. For a complex Banach space $(B, \|\cdot\|)$, let $\mathcal{H}(B)$ denotes the space of all entire functions on B. For each $m > 0$ we denote by $\operatorname{Exp}(B, \theta, m)$ to be

$$\operatorname{Exp}(B, \theta, m) = \left\{f \in \mathcal{H}(B);\ \ \|f\|_{\theta,m} := \sup_{z\in B}|f(z)|e^{-\theta(m\|z\|)} < \infty\right\}.$$

The space $\mathcal{F}_\theta(E^*_{\mathbb{C}})$ is defined by

$$\mathcal{F}_\theta(E^*_{\mathbb{C}}) = \operatorname{projlim}_{p\to\infty; m\downarrow 0} \operatorname{Exp}(E_{\mathbb{C},-p}, \theta, m). \tag{6}$$

In the remainder of this paper we simply use $\mathcal{F}_\theta$ for $\mathcal{F}_\theta(E^*_{\mathbb{C}})$. It is noteworthy that, for each $\xi \in E_{\mathbb{C}}$, the exponential function $e_\xi(z) := e^{\langle z,\xi\rangle}$, $z \in E^*_{\mathbb{C}}$, belongs to $\mathcal{F}_\theta$ and the set of such test functions spans a dense subspace of $\mathcal{F}_\theta$. The space of linear continuous operators from $\mathcal{F}_\theta$ into its topological dual space $\mathcal{F}^*_\theta$ is denoted by $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}^*_\theta)$ and assumed to carry the bounded convergence topology. Let μ be the standard Gaussian measure on E^* uniquely specified by its characteristic function

$$e^{-\frac{1}{2}|\xi|_0^2} = \int_{E^*} e^{i\langle x,\xi\rangle}\mu(dx), \quad \xi \in E.$$

Under condition (5), we have the nuclear Gel'fand triple (2), see Ref. [5].

2.2. QWN-derivatives

For $z \in E^*_{\mathbb{C}}$ and $\varphi(x)$ with Taylor expansion $\sum_{n=0}^\infty \langle x^{\otimes n}, f_n\rangle$ in $\mathcal{F}_\theta$, the holomorphic derivative of φ at $x \in E^*_{\mathbb{C}}$ in the direction z is defined by $(a(z)\varphi)(x) := \lim_{\lambda\to 0}\frac{\varphi(x+\lambda z)-\varphi(x)}{\lambda}$. We can check that the limit always exists, $a(z) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$ and $a^*(z) \in \mathcal{L}(\mathcal{F}^*_\theta, \mathcal{F}^*_\theta)$, where $a^*(z)$ is the adjoint of $a(z)$. For $\zeta \in E_{\mathbb{C}}$, $a(\zeta)$ extends to a continuous linear operator from $\mathcal{F}^*_\theta$ into itself (denoted by the same symbol) and $a^*(\zeta)$ (restricted to $\mathcal{F}_\theta$) is a continuous linear operator from $\mathcal{F}_\theta$ into itself. If $z = \delta_t \in E^*_{\mathbb{C}}$ we simply write a_t instead of $a(\delta_t)$. In QWN-field theory a_t and a^*_t are called the annihilation and creation operators at the point $t \in \mathbb{R}$.

The symbol and the Wick symbol, denoted by σ and ω respectively, of $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}^*_\theta)$ are by definition [14] the $\mathbb{C}$-valued function on $E_{\mathbb{C}} \times E_{\mathbb{C}}$ obtained as

$$\sigma(\Xi)(\xi,\eta) = \langle\langle \Xi e_\xi, e_\eta\rangle\rangle,$$

$$\omega(\Xi)(\xi,\eta) = \langle\langle \Xi e_\xi, e_\eta\rangle\rangle e^{-\langle\xi,\eta\rangle}, \quad \xi, \eta \in E_{\mathbb{C}}, \tag{7}$$

respectively, where $\langle\langle\cdot,\ \cdot\rangle\rangle$ denotes the duality between the two spaces $\mathcal{F}^*_\theta$ and $\mathcal{F}_\theta$.

It is a fundamental fact in quantum white noise theory [14] (see also Ref. [10]) that every white noise operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}^*_\theta)$ admits a unique Fock expansion (3) where, for each pairing $l,\ m \geq 0$, $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*_{sym(l,m)}$ and $\Xi_{l,\ m}(\kappa_{l,\ m})$ is the integral kernel operator characterized via the Wick symbol transform by

$$\omega(\Xi_{l,m}(\kappa_{l,m}))(\xi,\eta) = \langle\kappa_{l,m}, \eta^{\otimes l}\otimes\xi^{\otimes m}\rangle, \quad \xi, \eta \in E_{\mathbb{C}}. \tag{8}$$

This can be formally rewritten as (4). For $\zeta \in E_{\mathbb{C}}$, the *quantum white noise derivatives* are defined by

$$D^+_\zeta\Xi = [a(\zeta), \Xi], \quad D^-_\zeta\Xi = -[a^*(\zeta), \Xi]. \tag{9}$$

These are called the *creation derivative* and *annihilation derivative* of Ξ, respectively. For $z \in E^*_{\mathbb{C}}$, D^+_z is a continuous operator from $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$ into itself and D^-_z is a continuous operator from $\mathcal{L}(\mathcal{F}^*_\theta, \mathcal{F}^*_\theta)$ into itself. The pointwisely quantum white noise derivatives $D^\pm_t \equiv D^\pm_{\delta_t}$ are discussed in Ref. [9].

2.3. Basic topological isomorphisms

For $p \in \mathbb{E}_{\mathbb{C}}$ and $\gamma_1, \gamma_2 > 0$, we define the Hilbert spaces

$$F_{\theta,\gamma_1,\gamma_2}(E_{\mathbb{C},p}\oplus E_{\mathbb{C},p}) = \Big\{\overrightarrow{\varphi} = (\varphi_{l,m})_{l,m=0}^\infty;\ \varphi_{l,m} \in \times (E_{\mathbb{C},p}^{\otimes l}\otimes E_{\mathbb{C},p}^{\otimes m})_{sym(l,m)},\ \|\overrightarrow{\varphi}\|^2_{\theta,p,(\gamma_1,\gamma_2)} < \infty\Big\},$$

where

$$\|\overrightarrow{\varphi}\|^2_{\theta,p,(\gamma_1,\gamma_2)} := \sum_{l,m=0}^\infty (\theta_l\theta_m)^{-2}\gamma_1^{-l}\gamma_2^{-m}|\varphi_{l,m}|_p^2$$

and $\theta_n = \inf_{r>0} e^{\theta(r)}/r^n, > n \in \mathbb{E}_{\mathbb{C}}$. Put $F_\theta(E_{\mathbb{C}}\oplus E_{\mathbb{C}}) = \bigcap_{p\in\mathbb{E}_{\mathbb{C}},\gamma_1,\gamma_2>0} F_{\theta,\gamma_1,\gamma_2}(E_{\mathbb{C},p}\oplus E_{\mathbb{C},p})$. Let $\mathcal{H}_\theta(E_{\mathbb{C}}\oplus E_{\mathbb{C}}) =$