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Two conservative difference schemes for a model of nonlinear dispersive equations

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ABSTRACT

Two conservative differences schemes for the nonlinear dispersive Benjamin–Bona–Mahony–KdV (BBM-KdV) equation are proposed. The first scheme is two-level and nonlinear-implicit. The second scheme is three-level and linear implicit. Existence of its difference solutions has been shown. It is proved by the discrete energy method that the two schemes are uniquely solvable, unconditionally stable and the convergence is of second-order in the maximum norm. An iterative algorithm is proposed for solving the nonlinear scheme. The particular case known as the RLW equation is also discussed numerically in detail. Furthermore, three invariants of motion are evaluated to determine the conservation properties of the problem. Interaction of solitary waves with different amplitudes are shown. The three invariants of the motion are evaluated to determine the conservation proprieties of the system. The temporal evaluation of a Maxwellian initial pulse is then studied. Some numerical examples are given in order to validate the theoretical results.

1. Introduction

The Benjamin–Bona–Mahony (BBM) equation is also called the regularized long wave (RLW) equation

$$u_t - u_{xxt} + u_x + uu_x = 0, \qquad (1.1)$$

and its counterpart, the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + u_x + uu_x = 0, \qquad (1.2)$$

have both been proposed as models for long waves in nonlinear dispersive systems [1,2]. Both equations have a family of stable solitary-wave solutions of permanent form. Because the analytical solution of the BBM equation is not very useful, the availability of accurate and efficient numerical methods is essential. Numerical solutions of the BBM equation have been undertaken by employing various form of the finite difference methods [3–7], finite element methods [8–11], homotopy perturbation method [12,13], He's variational iteration method [14], the first integral method for the modified RLW equation [15], and other numerical techniques (see [16–22] and references therein).

In this work, we establish two conservative difference schemes for a model of nonlinear dispersive equations typified by the BBM–KdV equation and prove that the schemes are unconditionally stable and convergent with the convergence of $O(h^2+k^2)$ in the discrete L^∞-norm. Therefore, we shall approximate numerically, by a periodic code, solutions of the initial-value problem that decay to zero as $|x| \to \infty$. Such solutions include the solitary waves of BBM–KdV equation on the real line and by products of their interactions. Hence, by taking the interval $[0, L]$ large enough in each numerical experiment, so that the solution remains sufficiently small at the endpoints, we can approximate in a satisfactory manner the generation and interaction of real line solitary waves for finite time intervals.

We consider the initial-value and L-periodic boundary-value problems for the BBM–KdV equation; thus we seek a real-valued function $u(x, t)$, L-periodic in x, that satisfies

$$u_t - u_{xxt} + \alpha u_{xxx} + u_x + \beta uu_x = 0, \quad (x,t) \in \Omega \times [0,T], \qquad (1.3)$$

$$u(x,0) = u_0(x), \quad x \in \bar{\Omega}, \qquad (1.4)$$

where u_0 is a L-given periodic real function, $\Omega = [0, L]$, $0 < T < \infty$, α and β are given real constants.

It is easy to verify that problem (1.3)–(1.4) satisfies the following conservative laws:

$$Q(t) = \int_0^L u(x,t)dx = \int_0^L u_0(x)dx = Q(0). \qquad (1.5)$$

$$E(t) = ||u(\cdot,t)||_{L^2}^2 + ||u_x(\cdot,t)||_{L^2}^2 = ||u_0||_{L^2}^2 + ||(u_0)_x||_{L^2}^2 = E(0).$$

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(1.6)

The remainder of the article is arranged as follows: In Section 2, a nonlinear conservative difference scheme is derived. Conservation of discrete mass and discrete energy are discussed, a priori estimates for numerical solutions, existence, uniqueness and convergence for the difference scheme are proved. In Section 3 an iterative algorithm for the difference solution is given and its convergence is also proved. In Section 4, the linear scheme is presented and its priori estimates and L^∞-convergence are shown. In the last section, some numerical experiments are presented to support our theoretical results.

Throughout this article, C denotes a generic positive constant which is independent of the discretization parameters h and k, but may have different values at different places.

2. A nonlinear conservative difference scheme

In this section, we propose a two-level nonlinear Crank–Nicolson type finite difference scheme for the problem (1.3)–(1.4). For convenience, the following notations are used. For a positive integer N, let time-step $k = \frac{T}{N}$, $t_n = nk$, $n = 0, 1, \ldots, N$. Let space-step $h = \frac{L}{J}$, $x_j = jh$, $j = 0, \ldots, J$, and let

$$\mathbb{R}^J_{per} = \{V = (V_j)_{j\in\mathbb{Z}} \ / \ V_j \in \mathbb{R} \text{ and } V_{j+J} = V_j, \quad j \in \mathbb{Z}\}.$$

For a function $V^n \in \mathbb{R}^J_{per}$, define the difference operators as:

$$(V^n_j)_x = \frac{V^n_{j+1} - V^n_j}{h}, \quad (V^n_j)_{\bar{x}} = \frac{V^n_j - V^n_{j-1}}{h}, \quad (V^n_j)_{\hat{x}} = \frac{V^n_{j+1} - V^n_{j-1}}{2h},$$

$$(V^n_j)_t = \frac{V^{n+1}_j - V^n_j}{k}, \quad (V^n_j)_{\hat{t}} = \frac{V^{n+1}_j - V^{n-1}_j}{2k},$$

$$V^{n+\frac{1}{2}}_j = \frac{V^{n+1}_j + V^n_j}{2}, \quad \bar{V_j}^n = \frac{V^{n+1}_j + V^{n-1}_j}{2}.$$

For any function $V^n, W^n \in \mathbb{R}^J_{per}$, we introduce the discrete L^2 inner product in $\mathbb{R}^J_{per}$

$$\langle V^n, W^n\rangle = h\sum_{j=1}^{J} V^n_j W^n_j,$$

and Sobolev norms (or seminorms)

$$||V^n|| = \sqrt{h\sum_{j=1}^{J} V^2_j}, \quad ||V^n_x|| = \sqrt{h\sum_{j=1}^{J} (V^n_j)^2_x},$$

$$||V^n_{\hat{x}}|| = \sqrt{h\sum_{j=1}^{J} (V^n_j)^2_{\hat{x}}}, \quad ||V^n||_\infty = \max_{1\le j\le J} |V^n_j|.$$

Denote as $H^m_{per}(\Omega)$ the periodic Sobolev space of order m.

We discretize the problem (1.3)–(1.4) by the following Crank–Nicolson type difference scheme. We approximate $u^n_j \in \mathbb{R}^J_{per}$, $u^n_j := u(x_j, t^n)$, by $U^n_j \in \mathbb{R}^J_{per}$.

$$(U^n_j)_t - (U^n_j)_{x\bar{x}t} + \alpha\left(U^{n+\frac{1}{2}}_j\right)_{x\bar{x}\hat{x}} + \left(U^{n+\frac{1}{2}}_j\right)_{\hat{x}} + \varphi\left(U^{n+\frac{1}{2}}_j, U^{n+\frac{1}{2}}_j\right) = 0$$
$$j = 1, \ldots, J, \quad n = 0, \ldots, N-1, \tag{2.1}$$

$$U^n_j = U^n_{j+J}, \quad j = 1, \ldots, J, \quad n = 0, \ldots, N, \tag{2.2}$$

$$U^0_j = u_0(x_j), \quad j = 1, \ldots, J, \tag{2.3}$$

where

$$\varphi\left(U^{n+\frac{1}{2}}_j, U^{n+\frac{1}{2}}_j\right) = \frac{\beta}{3}\left(\left(U^{n+\frac{1}{2}}_j\right)\left(U^{n+\frac{1}{2}}_j\right)_{\hat{x}} + \left[\left(U^{n+\frac{1}{2}}_j\right)^2\right]_{\hat{x}}\right).$$

Next, we need the following lemma, which is a consequence of the well-known formulas of summations by parts and related periodicity conditions.

Lemma 1. *For any grid functions $V, W \in \mathbb{R}^J_{per}$ we have*

$$\langle V_x, W\rangle = -\langle V, W_{\bar{x}}\rangle, \tag{2.4}$$

$$\langle V_{\hat{x}}, W\rangle = -\langle V, W_{\hat{x}}\rangle, \tag{2.5}$$

$$\langle V_{x\bar{x}}, V\rangle = -||V_x||^2, \tag{2.6}$$

$$\langle V_{\hat{x}}, V\rangle = 0, \tag{2.7}$$

$$\langle V_{x\bar{x}\hat{x}}, V\rangle = 0, \tag{2.8}$$

$$\langle \varphi(V, V), V\rangle = 0. \tag{2.9}$$

Lemma 2. *For any discrete function $V \in \mathbb{R}^J_{per}$, we have*

$$||V_{\hat{x}}||^2 \le ||V_x||^2. \tag{2.10}$$

Proof. For $V \in \mathbb{R}^J_{per}$, we have

$$||V_x||^2 = ||V_{\bar{x}}||^2. \tag{2.11}$$

By Cauchy Schwartz inequality, we obtain for $V \in \mathbb{R}^J_{per}$

$$\begin{aligned}||V_{\hat{x}}||^2 &= h\sum_{j=1}^{J}\left(\frac{V_{j+1} - V_{j-1}}{2h}\right)^2 \\ &= h\sum_{j=1}^{J}\left(\frac{V_{j+1} - V_j}{2h} + \frac{V_j - V_{j-1}}{2h}\right)^2 \\ &\le 2h\sum_{j=1}^{J}\left(\frac{V_{j+1} - V_j}{2h}\right)^2 + 2h\sum_{j=1}^{J}\left(\frac{V_j - V_{j-1}}{2h}\right)^2 \\ &= \frac{1}{2}||V_x||^2 + \frac{1}{2}||V_{\bar{x}}||^2.\end{aligned}$$

Therefore, the inequality (2.10) can be easily proved from (2.11). □

Lemma 3. *For any two mesh functions $X, Y \in \mathbb{R}^J_{per}$, if $||X||_\infty$ is bounded then there exists a positive constant C such that:*

$$\Big| < \varphi(X, X) - \varphi(Y, Y), X - Y > \Big| \le C\Big(||X - Y||^2 + ||X - Y||^2_{\hat{x}}\Big).$$

Proof. Set $Z = X - Y$, we have $X^2 - Y^2 = (X - Y)(X + Y) = 2XZ - Z^2$.

Noting

$$< (X^2)_{\hat{x}} - (Y^2)_{\hat{x}}, Z >= 2 < (XZ)_{\hat{x}}, Z > - < (Z^2)_{\hat{x}}, Z >,$$

and

$$\begin{aligned}< X(X)_{\hat{x}} - Y(Y)_{\hat{x}}, Z > &= - < Z(Z)_{\hat{x}}, Z > + < Z(X)_{\hat{x}}, Z > \\ &\quad + < X(Z)_{\hat{x}}, Z > .\end{aligned}$$

Consequently

$$\begin{aligned}< \varphi(X, X) - \varphi(Y, Y), Z > &= - < \varphi(Z, Z), Z > -\frac{\beta}{3} < X(Z)_{\hat{x}}, \\ &\quad Z > +\frac{\beta}{3} < Z(X)_{\hat{x}}, Z > .\end{aligned} \tag{2.12}$$

Since $||X||_\infty$ is bounded, then there exists constant C such that

$$- < X(Z)_{\hat{x}}, Z > \le C||Z_{\hat{x}}||\,.\,||Z||. \tag{2.13}$$