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Operator-based approach for the construction of analytical soliton solutions to nonlinear fractional-order differential equations

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ABSTRACT

An operator-based framework for the construction of analytical soliton solutions to fractional differential equations is presented in this paper. Fractional differential equations are mapped from Caputo algebra to Riemann-Liouville algebra in order to preserve the additivity of base function powers under multiplication. The proposed technique is used for the construction of solutions to a class of fractional Riccati equations. Recurrence relations between power series parameters yield generating functions which are used to construct explicit expressions of closed-form solutions.

1. Introduction

Even though the idea of fractional differentiation dates back to Leibniz's letter to L'Hopital at the end of the 17th century [1], topics related to fractional derivatives and fractional differential equations have recently become the focus of vigorous research. The reason for renewed attention to this field has been due to the discovery of applications of fractional differential equations in many areas of science and engineering [2,3]. A comprehensive review on the application of fractional calculus in physics can be found in [4]. Numerical methods for fractional differential equations are considered in [5]. Examples of practical applications of fractional calculus are given in [6].

Fractional differential equations play a central role in viscoelasticity research, which was prompted by the wide use of polymer materials in engineering applications [7]. Fractional derivatives enable the modeling of materials that have memory of their previous deformations [8,9]. It is shown in [10] that models based on fractional derivative can be used to approximate the behavior of materials in Magnetic Resonance Elastography. A model of viscoelasticity employing fractional order derivatives is incorporated into Maxwell's equations in [11]. Fractional order controllers of a hexapod robot with viscous friction leg joints is considered in [12]. A particle-tracking approach using fractional differential equations is applied to simulate fractional diffusion-reaction processes in [13].

The introduction of fractional derivatives to the harmonic oscillator has led to new oscillatory phenomena [14]. An interpretation of fractional oscillators as an ensemble average of ordinary harmonic oscillators governed by a stochastic time arrow is given in [15]. The frequency and amplitude of the fractional-order Duffing oscillator are evaluated using a describing function method in [16]. The feedback control system is used as an active control strategy to control horseshoes chaos in a driven Rayleigh oscillator with fractional deflection in [17]. The eigenvalue spectrum of a fractional quantum harmonic oscillator is numerically investigated in [18].

Fractional derivatives are one of the novel concepts in recent advances of biomedical research [19]. In [20], the order of a fractional derivative is used to model two-stage human memory phenomena. It is demonstrated in [21] that nonlinear fractional-order models can be used to study valproic acid pharmacokinetics. The effect of fractional derivative order on a delayed predator-prey system with harvesting terms is considered in [22]. Lung parenchyma viscoelasticity is studied using fractional-order models in [23]. A

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fractional calculus approach is used to describe the dynamic behavior of cartilage in [24].

A number of new techniques for the construction of solutions to fractional differential equations have recently been developed. The waveform relaxation method is used to solve fractional differential-algebraic equations that arise in integrated circuits with new memory materials [25]. A spectral decomposition is used in conjunction with Fourier and Laplace transforms to solve the time-fractional diffusion equation in [26]. Artificial neural networks are used to construct numerical solutions to a number of different fractional differential equations in [27]. Blow-up solutions to the nonlinear fractional Schrödinger equation are studied using variational arguments and profile decomposition theory in [28].

Solutions to fractional wave-diffusion equations, modified anomalous fractional sub-diffusion equations and time-fractional telegraph equations are constructed using semi-analytical techniques based on the Fourier series expansion in [29]. Laplace and Fourier transforms with respect to both time and space variables are used to obtain fundamental solutions to the Riesz fractional advection-dispersion equation in [30]. Operational calculus of the Misukiński type is used to solve initial value problems on fractional differential equations with generalized Riemann-Liouville derivatives in [31]. Three time-splitting schemes for nonlinear time-fractional differential equations with smooth solutions are proposed in [32]. Second order implicit-explicit time-stepping schemes for nonlinear fractional differential equations with nonsmooth solutions are considered in [33]. The fractional-order Legendre operational matrix is used to construct approximate solutions to the fractional Riccati equation in [34].

A number of novel definitions of fractional derivatives have recently been introduced. Generalized fractional operators are applied to a particular class of nonstandard Lagrangians in [35]. A mixed integro-differential operator of the Erdélyi-Kober type that generalizes Riemann-Liouville and Caputo derivatives is introduced in [36]. Fractional calculus of variations based on the extended Erdélyi-Kober fractional integral is constructed in [37].

The main objective of this paper is to present an operator-based framework for the construction of closed-form analytical solutions to fractional differential equations. To better illustrate this new approach, we analyse one of the simplest nonlinear fractional-order models – the Riccati equation:

$$\left(\sqrt{x}\right)^k \left({}^C\mathbf{D}^{1/2}\right)^k y = B_0 + B_1 y + B_2 y^2; \quad k = 1, 2, \ldots, \tag{1}$$

where ${}^C\mathbf{D}^{1/2}$ denotes the Caputo fractional derivative of order $\frac{1}{2}$; $B_0, B_1, B_2 \in \mathbb{R}$.

The Caputo fractional derivative has been selected for this study because of its wide range of applications. Futhermore, the selection of a classical fractional derivative enables to more clearly illustrate the presented technique for the construction of closed-form solutions to fractional differential equations.

To aid clarity of exposition of the presented method for the construction of analytical solutions to (1), only fractional derivatives of order $\frac{1}{2}$ are considered. Derivations and computations given in this paper can be generalized for fractional derivatives of rational order $\frac{m}{n}$.

2. Motivation for (1)

Let us consider the ordinary Riccati equation with constant coefficients:

$$\frac{dz}{dx} = B_0 + B_1 z + B_2 z^2, \quad z(x_0) = z_0; \quad x_0, z_0, B_0, B_1, B_2 \in \mathbb{R}. \tag{2}$$

It is well-known that all solutions to (2) are kink solitary solutions [38]:

$$z = z_2 \frac{\exp\left(B_2(z_1 - z_2)(x - x_0)\right) - \frac{z_1(z_0 - z_2)}{z_2(z_0 - z_1)}}{\exp\left(B_2(z_1 - z_2)(x - x_0)\right) - \frac{z_0 - z_2}{z_0 - z_1}}, \tag{3}$$

where $z_1, z_2 \in \mathbb{C}$ are roots of the polynomial $B_0 + B_1 z + B_2 z^2$.

Direct solution of (2) is not straightforward for many of analytical solution construction methods. To counter this a typical independent variable substitution is used:

$$t = \exp(\eta x), \quad z(x) = \widehat{z}(t); \tag{4}$$

This transforms (5) into the Riccati equation with variable coefficients:

$$\eta t \frac{d\widehat{z}}{dt} = B_0 + B_1 \widehat{z} + B_2 \widehat{z}^2. \tag{5}$$

The general solution to (5) is a meromorphic function. It can be directly constructed using operator methods [39,40]:

$$\widehat{z} = \frac{\alpha_1 t - \beta_1}{\alpha_0 t - \beta_0}, \quad \alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}. \tag{6}$$

Thus equation (1) can be considered as a generalization of (5) in respect of Caputo fractional derivatives.

3. Preliminaries: main concepts and definitions

In this section, main concepts and definitions of the approach used to construct solutions to fractional-order nonlinear differential equations are given.

3.1. Power series extension

Functions that are analyzed in this paper are represented by the following power series:

$$y(z) = \sum_{j=0}^{+\infty} a_j \frac{z^j}{j!}, \quad z, a_j \in \mathbb{C}. \tag{7}$$

Coefficients a_j are constructed using operator techniques. The following two cases with respect to convergence of (7) in the Cauchy sense are considered:

1. Series (7) converges for $|z| < R$; $R > 0$. Then (7) can be extended to a wider complex domain (that does not include the singularities of (7)) using classical extension techniques. Letting $x \in \mathbb{R}$ be the argument of this extended function yields a real-argument power series $y(x)$ that is defined for values of x not necessarily in the convergence radius $|x| < R$. In this case, the extended function $y(x)$ and its power series representation are considered congruent.
2. Series (7) converges only for $|z| = 0$. In this case, the series is divergent in the Cauchy sense. However, such series still do contain important information [41]. Thus (7) can be considered as structural solutions to fractional differential equations without seeking congruent extended functions.

3.2. Fractional power series

For any $x \geq 0$, the following base functions are defined:

$$z_j = z_j(x) := \frac{x^{\frac{j-1}{2}}}{\Gamma\left(\frac{j+1}{2}\right)}, \quad j = 0, 1, \ldots; \tag{8}$$

where Γ is the gamma function:

$$\Gamma(x) = \int_0^{+\infty} \xi^{x-1} \exp(-\xi)\, d\xi. \tag{9}$$