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Generalized Nonsmooth Saddle Point Theorem and its applications on second order Hamiltonian systems

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ABSTRACT

The Generalized Nonsmooth Saddle Point Theorem is proved, which generalizes the previous ones. As its application, we obtain the existence of nontrivial periodic bouncing solution for systems $\ddot{x} = f(t, x)$ with new sublinear conditions, which has physical background.

1. Introduction and main results

We consider the following second order Hamiltonian systems with an obstacle, that is,

$$\ddot{x} = f(t, x), \quad \text{if } t \in \mathbf{R} \setminus W, \tag{1.1}$$

associated with the conditions

$$\begin{cases} \dot{x}(t^-) = -\dot{x}(t^+), & \text{if } t \in W, \\ x(t) \geq 0, & \forall\, t \in \mathbf{R}, \\ x(t) = x(t+T), & \forall\, t \in \mathbf{R}, \end{cases} \tag{1.2}$$

where $W = \{t \in \mathbf{R} \mid x(t) = 0\}$, f: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is T-periodic in t, continuous for all t and $x \in \mathbf{R}$.

Definition 1.1 (see [13]). Continuous map x: $\mathbf{R} \to \mathbf{R}$ is a nontrivial periodic bouncing solution of system (1.1), if it satisfies (1.1), (1.2) and

(1) the set W is nonempty and discrete,
(2) there exists at least one $t_0 \in W$ such that $\dot{x}(t_0^-) \neq 0$.

We say that there is a *real impact* at $t_0 \in W$, if $\dot{x}(t_0^-) \neq 0$. We call systems having solutions satisfying Definition 1.1 as *impact Hamiltonian systems*. In mechanics, (1.1) and (1.2) mean that the particle moves in the positive half-axis $x \geq 0$ and bounces in a perfectly way when it hits the obstacle at equilibrium point $x = 0$.

Widely applied in physics and engineering, impact systems have been considered by numerous authors (see [1,3,11–13,16]) using topological method. But the main difficulty of this kind of problem is that one can not specify the zero set of a bouncing periodic solution of (1.1). To the authors' knowledge, in paper [8], Jiang firstly presented a variational approach (Symmetric Mountain Pass Lemma) to study the impact systems, and established the existence of a sequence of bouncing solutions for second order impact Hamiltonian systems under a classical superquadratic condition

$$f(t, x) \geq \theta F(t, x) > 0, \quad x \geq r > 0 \text{ (where constant } \theta > 2),$$

and

$$\left| \frac{\partial F(t, x)}{\partial t} \right| \leq C(1 + F(t, x)), \quad x \in [0, +\infty), \; C > 0,$$

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where $F(t,x) = \int_0^x f(t,s)\mathrm{d}s$. Then, in paper [4], Ding considered the existence of subharmonic bouncing solutions for system (1.1) with sublinear conditions, that is, f satisfies

(F_0) there exist two functions $g \in L^1([0,T];\mathbf{R}), \gamma \in L^1\big([0,T];[0,+\infty)\big)$ and a constant $\alpha \in [0,1)$ such that

$$\limsup_{|x|\to+\infty} |f(t,x)-g(t)|/|x|^\alpha \le \gamma(t), \tag{1.3}$$

for all $x\in\mathbf{R}$ and uniformly a.e. $t\in[0,T]$, and

$$F(t,x)/|x|^{2\alpha} \to +\infty \quad \text{as } |x| \to +\infty, \tag{1.4}$$

and F satisfies

$$\left|\frac{\partial F(t,x)}{\partial t}\right| \le \sigma_0 F(t,x), \quad \text{a.e. } t\in[0,T], \quad x\in[0,+\infty).$$

for some constant $\sigma_0 > 0$. In paper [5], Ding generalized his result in [4] by replacing condition (1.4) with

$$\liminf_{|x|\to+\infty} F(t,x)/|x|^{2\alpha} > k^2\|g\|^2_{L^1}/24, \qquad k\in\mathbf{N}^*, \quad \text{a.e. } t\in[0,2\pi].$$

Different from the methods in papers [4,5,8], we firstly prove a Generalized Nonsmooth Saddle Point Theorem, then apply it to find out nontrivial kT-periodic bouncing solutions for system (1.1) with another sublinear condition different from (1.3) and (1.4).

Define set $\Gamma = \{h \in C([0,+\infty),[0,+\infty)) \mid h \text{ satisfies } (h1)-(h4)\}$, where

$(h1)$ $h(s) \le h(t) + C$, for a certain constant $C > 0$ and $s,t\in[0,+\infty)$ with $s\le t$,

$(h2)$ $h(s+t) \le C^*\big(h(s)+h(t)\big)$, for a certain constant $C^* \ge 0$ and $\forall s,t\in[0,+\infty)$,

$(h3)$ $th(t) - 2H(t) \to -\infty$, *as* $t\to+\infty$,

$(h4)$ $H(t)/t^2 \to 0$, *as* $t\to+\infty$, and

$H(t) := \int_0^t h(s)\mathrm{d}s$. Indeed, $h(t) = t^\alpha (\alpha\in[0,1))$ and $h(t) = \ln(1+t)$ can be in Γ, thus $\Gamma\neq\emptyset$.

We suppose that function f satisfies the following conditions,

(f) there exist T-periodic functions $\gamma, g \in L^1\big([0,T],(0,+\infty)\big)$ and function $h\in\Gamma$ such that

$$|f(t,x)| \le \gamma(t)h(|x|) + g(t), \quad \forall x\in\mathbf{R} \text{ and } t\in[0,T]. \tag{1.5}$$

(F1) The function h comes from condition (f) satisfies

$$\limsup_{|x|\to+\infty} \frac{1}{H(|x|)}\int_0^T F(t,|x|)\mathrm{d}t < 0. \tag{1.6}$$

(F2) Function $f(\cdot,x)$ is differentiable for a.e. $t\in[0,T]$ and there exists a constant $\sigma>0$ such that

$$\left|\frac{\partial F(t,x)}{\partial t}\right| \le -\sigma F(t,x), \quad \text{a.e. } t\in[0,T] \text{ and } x\in[0,+\infty).$$

In this paper, we assume $\dot{x}(t_i^-) \le 0$, and $\dot{x}(t_i^-) < 0$ if t_i is a real impact. Let x be a kT-periodic nontrivial bouncing solution of system (1.1) with isolated zeros $\{t_i\}_{i=1}^n$ with $0 < t_1 < t_2 < \cdots < t_n < kT$. Integrate (1.1) on $[0,kT]$, one has

$$\int_0^{kT} \ddot{x}(t)\mathrm{d}t = \int_0^{kT} f(t,x(t))\mathrm{d}t,$$

while,

$$\int_0^{kT}\ddot{x}(t)\mathrm{d}t = \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \ddot{x}(t)\mathrm{d}t = \sum_{j=0}^n \left(\dot{x}(t_{j+1}^-) - \dot{x}(t_j^+)\right) = 2\sum_{j=1}^n \dot{x}(t_j^-),$$

where $t_0 = 0$ and $t_{n+1} = kT$. It follows that

$$\int_0^{kT} f(t,x(t))\mathrm{d}t = 2\sum_{j=1}^n \dot{x}(t_j^-). \tag{1.7}$$

If there is at least one real impact, (1.7) implies

$$\int_0^{kT} f(t,x(t))\mathrm{d}t < 0.$$

To ensure that there exists a real impact at least, we give the following condition (B),

(B) Function $f(t,x) \le 0$ holds for all $t\in[0,T]$ and $x\ge0$, furthermore, $\lim_{x\to+\infty} f(t,x) = -\infty$ or $\limsup_{x\to+\infty} f(t,x) < 0$ hold for $t\in[0,T]$.

Now we list our main result of periodic bouncing solution as following.

Theorem 1.1. *Suppose function f satisfies conditions (B), (f), (F1) and (F2), then system (1.1) possesses nontrivial kT-periodic bouncing solutions u_k for every sufficiently large integer k. Furthermore, $\|u_k\|_\infty \to +\infty$ as $k\to+\infty$.*

There are functions f (see Example 4.1 in Section 4) satisfying our theorem but dissatisfying the conditions in [4].

Remark 1.1. In this paper, we replace the control term $|x|$ of f with a more general function $h(|x|)$. Actually, condition (1.3) is a special case of (1.5), if $h(t) = t^\alpha(\alpha\in[0,1))$. Moreover, our condition (1.6) is different from condition (1.4).

Remark 1.2. With our conditions (1.5) and (1.6), we can not use the same methods used in papers [4,5,8] to prove $\|x_k\|_\infty \to +\infty$ as $k\to+\infty$. Therefore, we firstly prove a Generalized Nonsmooth Saddle Point Theorem in Section 2, using the good property of which, then we can overcome the difficulty (see Lemma 3.6).

By the same analysis as in paper [8], we know that, if x: $\mathbf{R}\to\mathbf{R}$ is a kT-periodic solution with isolated zeros of

$$\ddot{x} = f(t,|x|)\mathrm{sgn}(x), \tag{1.8}$$

then $|x|$ is a nontrivial kT-periodic bouncing solution of system (1.1), where $\mathrm{sgn}(x)$ is defined as $\mathrm{sgn}(x) = \begin{cases} x/|x|, & x\neq0, \\ 0, & x=0. \end{cases}$

Our paper is organized as follows. In Section 2, we recall some notions about locally Lipschitzian functionals from [2] and [7], and give a Generalized Nonsmooth Saddle Point Theorem and some preliminaries. In Section 3, we give the proof of Theorem 1.1. Based on above analysis, we need two steps to prove Theorem 1.1. Paper [8] tells us that the corresponding functional of (1.8) is only locally Lipschitzian. So we firstly use our Generalized Nonsmooth Saddle Point Theorem in Section 2 to obtain kT-periodic solution x_k for equation (1.8). Then, with assumption (B), we prove the zero set $W_k = \{t\in\mathbf{R} \mid x_k(t) = 0\}$ is nonempty and the points in W_k are indeed isolated. In Section 4, an example is given to illustrate our Theorem 1.1.

2. Generalized nonsmooth saddle point theorem and preliminaries

Firstly, we recall some notions for locally Lipschitzian functionals. We refer to papers [2,7] for details.

Let E^* be the dual space of E. The generalized direction derivative of functional φ at $x_0\in E$ in the direction of $v\in E$ is defined by

$$\varphi^0(x_0;v) = \limsup_{h\to0,t\to0^+} \frac{\varphi(x_0+h+tv)-\varphi(x_0+h)}{t},$$

and the functional $v\to\varphi^0(x_0;v)$ is subadditive, positively homogeneous, convex and continuous. The Clarke generalized gradient $\partial\varphi(x_0)$ of φ at x_0 is

$$\partial\varphi(x_0) = \{w\in E^* \mid \langle w,v\rangle \le \varphi^0(x_0,v), \forall\, v\in E\},$$

which is a nonempty, convex, and weak*-compact subset of E^*. A point $x_0\in E$ is said to be a critical point of φ, if $\mathbf{0}\in\partial\varphi(x_0)$. In case