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Pattern formations of an epidemic model with Allee effect and time delay

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ABSTRACT

Allee effect widely exists for endangered plants and animals in ecosystem, which indicates that the minimum population density or size is necessary for population survival, namely, Allee threshold. In this paper, a delayed reaction-diffusion epidemic model with respect to Allee effect is investigated. The instability of the positive constant steady state is induced by two mechanisms, one is diffusion-induced instability, the other is delay-induced instability. The first case gives rise to Turing patterns. Moreover, Turing region becomes narrow as incubation delay being increased. We further observe that the range of Turing mode is enlarged with the increase of Allee threshold. The numerical simulations verify our theoretical results. The combined effects of Allee effect and disease on the spatial distributions of endangered species are studied, which provides new insights for human intervention in conservation management of these species.

1. Introduction

There are emergences of new epidemics and outbreaks of old epidemics each year, these diseases are threatening to human health and life, and diseases occurring in animals have some negative effects on social economy and ecological environment, such as Ebola, dengue, H7N9 avian influenza, rabies and so on [1–6]. From the Kermack–McKendric compartment epidemic model, the population is divided into different disease state compartments ($S, E, I, R, etc.$) to study the spread of the disease, and reasonable models are established based on propagation mechanism of specific diseases, which has a long history [7–12]. These results provide some theoretical guidance for the prevention and control of diseases.

The time scale of the sudden outbreak disease is relatively short, thus effect of demographic process can be ignored, while some diseases in human and animal populations have existed in a few years, decades, or even longer, which form endemic diseases, such as HIV, hepatitis B, brucellosis. Therefore, it is necessary to consider the joint effects of disease and demographic factor. The demographic process is mainly birth and death processes of population, which is from Malthus growth, Logistic growth to Allee effect. The Malthus growth model is that the number of the population increases infinitely in the form of exponential, which may be consistent with the initial growth of population [13]. However, in the actual situation, the individuals carry out competition for the limited living space and resources when the total number of population increases certain value, we thus need to subtract competition term among individuals in the Malthus growth model, which is known as Logistic growth model proposed by Verhuist in 1838 [14]. Through the study of a large number of biological populations, many phenomena show the positive relationship between population density and population growth rate, which reflects that population growth rate increases with the increase of population density in the dynamical evolution process of population when the population density is very low. But, the growth rate of the population will become negative and eventually extinct as the population density or size is lower than a critical value. Thus, the minimum population density or size is necessary for population survival, and

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this critical level is generally referred to as the Allee threshold. This idea was firstly proposed by ecologist Allee in 1931 [15]. For example, *Lycaon pictus* is a typically social animal, they have cooperation behaviors for breeding group, foraging food, fighting natural enemy, etc. A small population size leads to the loss of some cooperative skills, then will eventually become extinct [16,17]. The group with strong Allee effect will give rise to negative growth rate and eventually lead to extinction when the population density is too sparse, while the growth rate of the group still remain positive for such case, which indicates the group with week Allee effect. In general, these Allee effect mechanisms arise from cooperation or facilitation among individuals in the species. The Allee effect is ubiquitous for endangered plants and animals in ecosystem, therefore, it is reasonable to introduce demographic Allee effect into the epidemic model. The combined interplay of strong Allee effect and epidemic is investigated based on mathematical models [18–22].

The spread of disease in pervious papers do not consider the interactions among individuals depend on spatial environment [24–28]. In fact, the population lives in spatial environment for survival and reproduction. In order to look for the suitable living environment, they move to other space positions. In addition, the individuals hunt for food, search for mating partners or avoid natural enemy by diffusion. Based on these practical circumstances, the diffusion of the population in space should be considered. Moreover, it is necessary to study the spatial evolution of infectious disease so as to better reflect the propagation process of diseases. At present, there are many methods used to study the spatial structure of infectious diseases, such as cellular automata, metapopulation, reaction-diffusion equation and integro differential equations [29–36]. Moreover, the spatial distributions of susceptible and infectious individuals are obtained by analyzing Turing instability of reaction-diffusion equations, which is helpful to control the infectious diseases. The vast majority of diseases have incubation period, for example, the latent period of brucellosis is 2–3 weeks [3]. The susceptible individual infected need to take some time to show symptoms, therefore, it is reasonable to introduce incubation delay in reaction-diffusion epidemic model.

The structure of this paper is as follow. In Section 2, a delayed reaction-diffusion epidemic model with Allee effect is given, then the model is simplified by introducing dimensionless transformation. Moreover, the local stability of positive equilibria is analyzed. In Section 3, through assuming small delay, the characteristic equation of the delayed reaction-diffusion epidemic model with Allee effect is derived, and the instability of the positive equilibria caused by two different mechanisms, one is diffusion-induced instability, the other is delay-induced instability. Moreover, the first case gives rise to Turing patterns when parameters are chosen from Turing region. The range of Turing mode increases as Allee threshold being increased, and is not affected by time delay. In Section 4, the numerical simulations show rich Turing patterns. Finally, the conclusion and discussion are given.

2. Mathematical modeling and analysis

Some species suffer from the joint effects of disease and Allee effect. And disease may accelerate the extinction of populations with strong Allee effect, the possible reason is that disease reduces viability of the individual in population, which leads to the population density below the Allee threshold. For example, carnivore diseases such as rabies and canine distemper are the main factor of extinction of the endangered species in the Serengeti [23]. Thus, it is very meaningful to study the combined effects between disease and Allee effect. Hilker et al. accounted for *SI* epidemic model with Allee effect [21], the total population N consists of the susceptible X and the infected Y. Assume that the infected reproduces susceptible offspring, that is, there is no vertical transmission in such epidemic model. The strong Allee effect is expressed by the intrinsic per capita growth rate of the population $r(N)=a(K_+-N)(N-K_-)$, where a is per capita growth rate, K_+ is the carrying capacity, K_- determines the minimum viable density without respect to the infectious diseases, which is Allee threshold, and we are interested in $0<K_-\ll K_+$ from a biological point of view. They further assume that $b(N)$ and $m(N)$ are density-dependent fertility and natural mortality functions, respectively, the corresponding function forms are

$$b(N)=a[-N^2+(K_++K_-+e)N+c],$$

$$m(N)=a(eN+K_+K_-+c),$$

where e, c represent the effect of density dependence and independence in the demographic function, respectively, and they have no effects on the expression $r(N)$, then the following epidemic model with strong Allee effects is given:

$$\begin{cases}\dfrac{dX}{dT}=b(N)N-\beta XY-m(N)X,\\ \dfrac{dY}{dT}=\beta XY-m(N)Y-\mu Y,\end{cases}\tag{1}$$

where β is transmission rate, μ represents disease-induced mortality. We mainly focus on the time evolution of the densities of total population and the infected, the above system is written as:

$$\begin{cases}\dfrac{dN}{dt}=a(-N^2+(K_++K_-+e)N+c)N-a(eN+K_+K_-+c)N-\mu Y,\\ \dfrac{dY}{dt}=\beta(N-Y)Y-a(eN+K_+K_-+c)Y-\mu Y.\end{cases}\tag{2}$$

Most diseases experience an incubation period, that is, the susceptible individuals infected need to take some time to show symptoms, such as rabies, brucellosis, canine distemper and so on. Therefore, it is necessary to introduce incubation period into transmission term. Assume that incubation period is fixed delay τ. The population lives in spatial environment and randomly move to other places. We consider that the population lives in a closed spatial domain. In order to depict the spatial diffusion process, $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ describes the usual Laplacian operator in two-dimensional space. D_1, D_2 denote the diffusion coefficients for the total population and the infected, respectively. Furthermore, we are interested in the self-organization of patterns, then nonzero initial condition and Neumann boundary conditions are chosen. Then the following delayed reaction-diffusion *SI* model with Allee effect is given by:

$$\begin{cases}\dfrac{\partial N}{\partial T}=a(-N^2+(K_++K_-+e)N+c)N-a(eN+K_+K_-+c)N\\ \qquad -\mu Y+D_1\nabla^2N,\\ \dfrac{\partial I}{\partial T}=\beta(N(T-\tau)-Y(T-\tau))Y(T-\tau)-a(eN+K_+K_-+c)Y\\ \qquad -\mu Y+D_2\nabla^2Y.\end{cases}\tag{3}$$

By introducing dimensionless and time scale transformations, let $P=\frac{N}{K_+}$, $I=\frac{Y}{K_+}$, $t=aeK_+T$, $r=\frac{K_+}{e}$, $u=\frac{K_-}{K_+}\in(0,1)$, $d=\frac{c}{eK_+}$, $\sigma=\frac{\beta}{ae}$, $\alpha=\frac{\mu}{aeK_+}$, $d_1=\frac{D_1}{aeK_+}$, $d_2=\frac{D_2}{aeK_+}$, $\bar{\tau}=aeK_+\tau$. one can obtain