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Global Hopf bifurcation of an SIRS epidemic model with age-dependent recovery[☆]

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ABSTRACT

In this paper, an age structured SIRS epidemic model with age of recovery is studied which allows the removed individuals to become susceptible again when they lose the protection property as the time goes. The age structured model is reformulated as an abstract non-densely defined Cauchy problem and the expression of the basic reproduction number, $\mathcal{R}_0$, is obtained. If $\mathcal{R}_0 < 1$, the model only has the disease-free steady state and it is global stability. If $\mathcal{R}_0 > 1$, besides the disease-free steady state the model also has an endemic steady state. By analyzing the associated characteristic equation, the existence of a local Hopf bifurcation is proved under certain conditions. Also, we considered the global continuation of the local Hopf bifurcation. Finally, some numerical simulations are carried out to illustrate the theoretical results.

1. Introduction

Infectious diseases have brought many troubles and burden to humanity for centuries. Various mathematical models have been used in the study of disease spread. Specially, the classical compartment epidemic models are usually described as ODEs, the spatial epidemics can be studied by use of PDEs [1]. Complex networks are useful in revealing the role of social contact and the disease-behavior dynamics [2–4] and age structured models are valuable when the disease involves the demography or some age distributions.

Age-structured model is an important tool in the study of epidemiology which can describe properly the dynamic process of the disease spread. Magal et al. studied an SIR epidemic model with infection age and developed Lyapunov functionals for the age-structured model [5]. Huang et al. studied the global dynamic behaviors of an HIV infection model with infection age by use of Lyapunov functionals [6]. McCluskey proposed and studied an SEI model with latency age and infection age [7]. The roles of latency age and vaccination age in an SVEIR epidemic model are considered in [8]. Relapse age [9] and the other types of age applied in epidemic models can also play more and more important role. Classical results on age structured models focus on the global stability analysis of solutions.

Since age-structured models can be regarded as abstract non-densely defined Cauchy problems [10], and a center manifold theory [11] and a Hopf bifurcation theorem [12] have been developed for non-densely defined Cauchy problems. The dynamics of age structured epidemic models can result in bifurcation behaviors, besides the global stability. For instance, Liu and Wang studied an age-structured compartmental pest-pathogen with infection age and obtained a local Hopf bifurcation [15]. It is worth noting that the center manifold theory and Hopf bifurcation of a specific age structured population model are developed and studied in [16] and the normal form results for the age structured population model has been established recently [13,14]. Besides, Liu et al. [17] investigated a class of predator-prey model with age structure and obtained Bogdanov–Takens bifurcation results.

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Notice that Hopf bifurcation behaviors often occur in epidemic models, for example, the local Hopf bifurcation [20,21] and the global Hopf bifurcation [18,19] are obtained in epidemic models with time delay. Especially, both the delayed SEIRS epidemic model in [18] and the delayed SIR epidemic model in [21] have involved temporary immunity which is very useful to study the disease dynamics. As we know, SIRS epidemic models are more suitable than SIS models and SIR models, if the removed individuals can acquire temporary immunity from the disease for some periods and then lose the protection to become the susceptible. Since age is considered as a continuous variable, these processes of the immune protection can be properly described by some age structured epidemic models. In the paper, based on the above discussion, we wonder what role does the immunity age play in an SIRS epidemic model.

To study the role of age-dependent immunity, we start from an equation of the form

$$\frac{\partial R(a,t)}{\partial a}+\frac{\partial R(a,t)}{\partial t}=-(\delta(a)+\mu)R(a,t), \tag{1.1}$$

with the boundary and initial conditions

$$R(0,t)=\gamma I(t), \quad R(a,0)=R_0(a)\in L^1_+((0,+\infty),\mathbb{R}).$$

In (1.1), $R(a,\ t)$ is the density of recovered individuals with respect to the age of recovery a at time t. It is assumed that the newly recovered individuals enter the recovered class $R(a,\ t)$ with recovery-age equal to zero. The parameter μ is the natural death rate, $\delta(a)$ is the age-dependent progression rate of the removed to the susceptible class. We assume $\delta(a)$ is a bounded general function of the recovery-age a. Under these basic assumptions, we consider the following age-structured SIRS model

$$\begin{cases}\dfrac{dS(t)}{dt}=\Lambda-\beta S(t)I(t)-\mu S(t)+\displaystyle\int_0^\infty \delta(a)R(a,t)da,\\ \dfrac{dI(t)}{dt}=\beta S(t)I(t)-(\mu+\nu+\gamma)I(t),\\ \dfrac{\partial R(a,t)}{\partial a}+\dfrac{\partial R(a,t)}{\partial t}=-(\delta(a)+\mu)R(a,t),\\ R(0,t)=\gamma I(t),\\ S(0)=S_0,\quad I(0)=I_0,\quad R(a,0)=R_0(a)\in L^1_+(0,+\infty).\end{cases} \tag{1.2}$$

In model (1.2), $S(t)$ and $I(t)$ respectively represents the density of suspectable individuals and the density of infectious individuals at time t. The parameter Λ is the recruitment rate, β is the infection rate from the suspectable class to the infected class, γ is the recovery rate, ν is the death rate deduced by disease. We further assume that the function $\delta(a)$ belongs to $L^\infty_+((0,+\infty),\mathbb{R})\setminus\{0_{L^\infty}\}$.

The main aim of this paper is to investigate the existence of the global Hopf bifurcation of model (1.2). The rest of this paper is organized as follows. In Section 2, we give some preliminary results of model (1.2). Then the local and global stability results of the disease-free steady state E^0 are proved in Section 3. In Section 4, the existence of local Hopf bifurcation is proved by using Hopf bifurcation theorem. The global bifurcation of the periodic solution is discussed in Section 5. Finally, in Section 6, a brief discussion and some numerical examples are presented.

2. Preliminary results

By setting

$$N(t)=S(t)+I(t)+\int_0^{+\infty}R(a,t)da,$$

we deduce from (1.2) that $N(t)$ satisfies the following ordinary differential equation:

$$N'(t)=\Lambda-\mu N(t),$$

and therefore $N(t)$ converges to Λ/μ as T tends to infinity. Denote

$$\Omega=\left\{(S,I,R)\in\mathbb{R}_+\times\mathbb{R}_+\times L^1_+((0,+\infty),\mathbb{R}):\right.$$
$$\left. S+I+\int_0^{+\infty}R(a,\cdot)da\le\frac{\Lambda}{\mu}\right\}. \tag{2.1}$$

Then Ω is the maximum positively invariant set of system (1.2) and we can study the dynamics of model (1.2) in the bounded set Ω.

In order to prove the main results of the paper, we need to reformulate system (1.2) as a Cauchy problem. We can rewrite system (1.2) as the following

$$\begin{cases}\dfrac{\partial R(a,t)}{\partial a}+\dfrac{\partial R(a,t)}{\partial t}=-(\delta(a)+\mu)R(a,t),\\ \dfrac{dV(t)}{dt}=-CV(t)+G\Big(V(t),\displaystyle\int_0^\infty\delta(a)R(a,t)da\Big),\\ R(0,t)=\gamma I(t),\\ R(a,0)=R_0(a)\in L^1_+((0,+\infty),\mathbb{R}),\\ V(0)=V_0\in\mathbb{R}^2,\end{cases} \tag{2.2}$$

where

$$G\Big(V(t),\int_0^\infty\delta(a)R(a,t)da\Big)=\begin{pmatrix}\Lambda-\beta S(t)I(t)+\int_0^\infty\delta(a)R(a,t)da\\ \beta S(t)I(t)\end{pmatrix},$$

$$C=\begin{pmatrix}\mu & 0\\ 0 & \mu+\nu+\gamma\end{pmatrix},\quad V(t)=\begin{pmatrix}S(t)\\ I(t)\end{pmatrix},$$

$$V(0)=V_0=\begin{pmatrix}S_0\\ I_0\end{pmatrix}\in\mathbb{R}^2.$$

In system (2.2), by setting

$$V(t):=\int_0^\infty V(a,t)da,$$

where $V(a,t)=\begin{pmatrix}V_1(a,t)\\ V_2(a,t)\end{pmatrix}$. We can rewrite the ordinary differential equation in (2.2) as an age-structured model

$$\begin{cases}\dfrac{\partial V(a,t)}{\partial a}+\dfrac{\partial V(a,t)}{\partial t}=-CV(a,t),\\ V(0,t)=G\Big(V(t),\displaystyle\int_0^\infty\delta(a)R(a,t)da\Big),\\ V(a,0)=V_0(a)\in L^1_+((0,+\infty),\mathbb{R}^2).\end{cases}$$

Thus by setting $w(a,t)=\begin{pmatrix}R(a,t)\\ V(a,t)\end{pmatrix}$, we obtain the following system

$$\begin{cases}\dfrac{\partial w(a,t)}{\partial a}+\dfrac{\partial w(a,t)}{\partial t}=-D(a)w(a,t),\\ w(0,t)=B(w(\cdot,t)),\\ w(a,0)=w_0(a)=\begin{pmatrix}R_0(a)\\ V_0(a)\end{pmatrix}\in L^1_+((0,+\infty),\mathbb{R}^3),\end{cases} \tag{2.3}$$