Chaos, Solitons and Fractals 103 (2017) 79–89

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

A priori estimates for weak solution for a time-fractional nonlinear reaction-diffusion equations with an integral condition

Oussaeif Taki-Eddine*, Bouziani Abdelfatah

Department of Mathematics and Informatics, The Larbi Ben M'hidi University, Oum El Bouaghi

ARTICLE INFO

Article history:
Received 19 March 2017
Revised 24 May 2017
Accepted 27 May 2017

Keywords:
Nonlinear fractional reaction-diffusion equations
Energy inequality
Integral condition
Existence
Uniqueness
Weak solution
Bouziani spaces

ABSTRACT

In this paper, we establish sufficient conditions for the existence and uniqueness of the solution in functional weighted Sobolev space for a class of initial-boundary value problems with integral condition for a class of nonlinear partial fractional reaction-diffusion (RD) equations. The results are established by using a priori estimate in Bouziani fractional spaces and applying an iterative process based on results obtained for the linear problem, we prove the existence, uniqueness of the weak generalized solution of the nonlinear problem.

1. Introduction

Nonlinear diffusion equations, an important class of parabolic equations, have come from a variety of diffusion phenomena which appear widely in nature. These are suggested as mathematical models of physical problems in many fields, such as filtration, phase transition, electromagnetism, acoustics, electrochemistry, cosmology, biochemistry and dynamics of biological groups. In many cases, the equations possess degeneracy or singularity, which makes the study more involved and challenging.

Fractional differential equations (FDEs) are obtained by generalizing differential equations to an arbitrary order. Since fractional differential equations are used to model complex phenomena, they play a crucial role in engineering, physics and applied mathematics. Therefore they have been generating increasing interest from engineers and scientist in recent years. Since FDEs have memory, nonlocal relations in space and time, complex phenomena can be modeled by using these equations. Due to this fact, the materials with memory and hereditary effects, and dynamical processes, including gas diffusion and heat conduction, in fractal porous media can be more adequately modeled by fractional-order models. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, signal processing, control theory, porous media, fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory and probability, signal processing, and many other physical processes are diverse applications of FDEs [1–5]. Recently, there has been a significant development in fractional differential and partial differential equations; see the monographs of Kilbas et al. [6] and in the papers [7–19] and the references therein.

However, many phenomena can better be described by integral boundary conditions. Integral boundary conditions are encountered in various applications such as population dynamics, blood-flow models, chemical engineering and cellular systems. Moreover boundary value problems with integral conditions originating from various engineering disciplines, is of growing interest. That is, a large number of physica phenomena and many problems in modern physics and technology can be described in terms of nonlocal problems, such as problems in partial differential equations with integral conditions. Which has been studied and received a great interest extensively by the author Bouziani in several works, among them [26–32].

The theory on existence and uniqueness of solutions of the initial and boundary value problems for fractional differential equations is extensivelystudied by many authors, see for example, [8–11,20,22,23] and references therein.

Especially, The study of existence and uniqueness of solution of fractional partial differential equations are then proved by the well-known Lax–Milgram theorem and fixed point theorem. Among them, we only mention here the papers [20,24,25].

* Corresponding author.
E-mail addresses: taki_maths@live.fr (O. Taki-Eddine), af_bouziani@hotmail.com (B. Abdelfatah).

http://dx.doi.org/10.1016/j.chaos.2017.05.035
0960-0779/

A suitable variational formulation is the starting point of many numerical methods, such as finite element methods and spectral methods. The existence and uniqueness of the variational solution is thus essential for these methods to be efficient. The construction of the variational formulation strongly relies on the choice of suitable spaces and norms. Motivated by this, we extend and generalize the study of the problems of partial differential equation with integral condition to the study of fractional partial differential equation with integral condition and expand the works in classical problem of fractional partial differential equation to non standard problem.

In recent time, interest in nonlinear fractional reaction-diffusion (RD) equations [35–38] has increased because the equation exhibits self organization phenomena and introduces a new parameter, the fractional index, into the equation. Additionally, the analysis of fractional RD equations is of great interest from the analytical and numerical point of view. From a mathematics point of view they also offer a rich and promising area of research. fractional RD equations have been investigated for certain boundary and initial conditions and in most cases explicit solutions cannot been found.

We present a study of solution of a more general model of nonlinear time-fractional RD equations:

$$D_t^\alpha u(x,t) = dD_x^2 u(x,t) + G(u);\ \ 0<\alpha\leqslant 1,\ \ x\in\mathbb{R} \text{ et } t>0.$$

where d is the diffusion coefficient and $G(u)$ is a nonlinear function representing reaction kinetics. Also, the time fractional Clannish Random Walker's Parabolic CRWP equation:

$$D_t^\alpha u(x,t) - D_x u(x,t) + D_x^2 u(x,t) = -2u(x,t)D_x u(x,t);\ 0 < \alpha\leqslant 1, x\in\mathbb{R} \text{ et } t>0.$$

The objective of this paper is to extend the application of the energy inequality method to obtain existence and uniqueness of weak solutions in functional weighted Sobolev space for a class of initial-boundary value problems with non local condition "integral condition" for a class of more general nonlinear partial fractional differential equations, which has not been studied so far.

This work explain and complete our article [39]. As well this work contains a new theorizing for new spaces: Bouziani fractional spaces.

2. Preliminaries

Definition 1. Let $\mathbf{\Gamma}(\cdot)$ denote the gamma function. For any positive integer $0<\sigma<1$, the Caputo derivative are the Riemann Liouville derivative are, respectively, defined as follows:

(i) The left Caputo derivatives:

$${}_0^C\partial_t^\sigma u(x,t) := \frac{1}{\mathbf{\Gamma}(1-\sigma)}\int_0^t \frac{\partial u(x,\tau)}{\partial\tau}\frac{1}{(t-\tau)^\sigma}d\tau. \tag{2.1}$$

(ii) The left Riemann–Liouville derivatives:

$${}_0^R\partial_t^\sigma u(x,t) := \frac{1}{\mathbf{\Gamma}(1-\sigma)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)}{(t-\tau)^\sigma}d\tau. \tag{2.2}$$

(iii) The right Riemann–Liouville derivatives:

$${}_t^R\partial_T^\sigma u(x,t) := \frac{(-1)}{\mathbf{\Gamma}(1-\sigma)}\frac{\partial}{\partial t}\int_t^T \frac{u(x,t)}{(t-\tau)^\sigma}d\tau. \tag{2.3}$$

Definition 2. Many authors think that the Caputo's version is more natural because it allows the handling of inhomogeneous initial conditions in an easier way. Then the two definitions (2.1) and (2.2) are linked by the following relationship, which can be verified by a direct calculation:

$${}_0^R\partial_t^\alpha u(x,t) = {}_0^C\partial_t^\alpha u(x,t) + \frac{u(x,0)}{\mathbf{\Gamma}(1-\alpha)t^\alpha}. \tag{2.4}$$

we denote the domain $\Omega=(0,1)\times(0,T)$, with $T<\infty$.

Definition 3 (Fractional Integration by parts:). If $v\in L^p(\Omega)$, $w\in L^q(\Omega)$ and $\frac{1}{p}+\frac{1}{q}\leqslant 1+\sigma$, then the following formula

$$\int_0^T v(t).\ ({}_0I_t^\sigma w)(t)\ dt = \int_0^T w(t).\ ({}_tI_T^\sigma v)(t)\ dt.$$

defines fractional integration by parts [33].

Definition 4. [21] For any real $0<\sigma<1$, we define the semi-norm: (the space ${}^RH_0^\sigma(\Omega)$ is denote by $H_l^\sigma(\Omega)$ in the article [2]).

$$|u|^2_{{}^RH_0^\sigma(\Omega)} := \left\|{}_0^R\partial_t^\sigma u\right\|^2_{L_2(\Omega)}, \tag{2.5}$$

and norm:

$$\|u\|_{{}^RH_0^\sigma(\Omega)} := \left(\|u\|^2_{L_2(\Omega)} + |u|^2_{{}^RH_0^\sigma(\Omega)}\right)^{\frac{1}{2}}, \tag{2.6}$$

we then define ${}^RH_0^\sigma(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{{}^RH_0^\sigma(\Omega)}$.

Definition 5. [21] For any real $0<\sigma<1$, we define the semi-norm: (the space ${}^RH_r^\sigma(\Omega)$ is denote by $H_r^\sigma(\Omega)$ in the article [2]).

$$|u|^2_{{}^RH_r^\sigma(\Omega)} := \left\|{}_t^R\partial_T^\sigma u\right\|^2_{L_2(\Omega)}, \tag{2.7}$$

and norm:

$$\|u\|_{{}^RH_r^\sigma(\Omega)} := \left(\|u\|^2_{L_2(\Omega)} + |u|^2_{{}^RH_r^\sigma(\Omega)}\right)^{\frac{1}{2}}, \tag{2.8}$$

we then define ${}^RH_r^\sigma(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{{}^RH_r^\sigma(\Omega)}$.

Remark 1. Firstly, we have $|u|_{{}^RH_0^\sigma(\Omega)}$ is a semi-norm and not a norm, because if we put:

$$u=(t-\tau)^\sigma(t-2\tau),$$

for $n=1$, we have

$$\begin{aligned}|u|_{{}^RH_0^\sigma(\Omega)} &= \left(\int_\Omega\left(\frac{1}{\mathbf{\Gamma}(1-\sigma)}\frac{\partial}{\partial t}\int_0^t\frac{[(t-\tau)^\sigma(t-2\tau)]}{(t-\tau)^\sigma}d\tau\right)^2 d\Omega\right)^{\frac{1}{2}}\\ &= \left(\int_\Omega\left(\frac{1}{\mathbf{\Gamma}(1-\sigma)}\frac{\partial}{\partial t}\int_0^t (t-2\tau)d\tau\right)^2 d\Omega\right)^{\frac{1}{2}}\\ &= 0.\end{aligned}$$

So we find $u\neq 0$, despite the fact that $|u|_{{}^RH_0^\sigma(\Omega)}=0$. Secondly, it suffices to apply the definition of a norm, and check the three essential properties.

Definition 6. [21] For any real $0<\sigma<1$, we define the semi-norm (the space $H_{l,r}^\sigma(\Omega)$ is denote by $H_c^\sigma(\Omega)$ in the article [21]):

$$|u|_{H_{l,r}^\sigma(\Omega)} := \left\langle {}_0^R\partial_t^\sigma u,\ {}_t^R\partial_T^\sigma u\right\rangle^{\frac{1}{2}}_{L_2(\Omega)}. \tag{2.9}$$

3. Bouziani functional spaces

3.1. Bouziani space

We introduce the function spaces, which we need in our investigation. $L_2(0,1)$ and $L_2(0,T,L_2(0,1))$ be the standard function spaces. We denote by $C_0(0,1)$ the vector space of continuous functions with compact support in (0.1). Since such functions are Lebesgue integrable with respect to dx, we can define on $C_0(0,1)$ the bilinear form given by

$$(u,w) = \int_0^1 \Im_x u\cdot\Im_x w\ dx, \tag{3.1}$$