Chaos, Solitons and Fractals 103 (2017) 123–130

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

A space-time fractional derivative model for European option pricing with transaction costs in fractal market☆

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ARTICLE INFO

Article history:
Received 5 March 2017
Revised 29 April 2017
Accepted 30 May 2017

Keywords:
Fractional differential equation
Option pricing
Approximate solution
Transaction cost

ABSTRACT

From the point of view of fractional calculus and fractional differential equation, the work handles European option pricing problems with transaction costs in fractal market. Under the definition of the modified Riemman-Liouville fractional derivative, the pricing model based on a space-time fractional patrial differential equation is presented by the replicating portfolio, containing the Hurst exponent taken as the order of fractional derivative. And then, European call and put options are constructed and calculated by the enhanced technique of Adomian decomposition method under the finite difference frame. The fractional derivative model is finally tested by the data from the option market.

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1. Introduction

Market friction exists in the real financial world. The existence of transaction costs relates the number of hedging and the price of options. The pricing models with transaction costs are important improvements for the classical Black-Scholes model. As early as 1985, Leland [1] gave a technique to replicate option returns in the presence of transaction costs. In 1992, Boyle and Vorst [2] took transaction costs into account and extended the Cox-Ross-Rubinstein binomial option pricing model. The model can be expressed by the Black-Scholes model with a modified volatility. Davis et al. [3] priced European options with proportional transaction charges based on a model similar to Black-Scholes one. With transaction costs, Barles and Soner [4] derived a nonlinear Black-Scholes equation with an adjusted volatility. Considering transaction costs and the risk from a volatile portfolio, Kratka derived a mathematical pricing model [5]. Afterwards, Jandačka and Ševčovič [6] extended the classical Black-Scholes equation and Lelands equation to a new model for pricing derivative securities under both transaction costs and the risk from the unprotected portfolio.

A large amount of researches have found that time series have long-range dependence and the market returns display scaling properties. That the financial market has fractal character is an important discovery and provides a new perspective for the theoretical researches of financial derivatives. In a fractal market, the fractional Black-Scholes models [7–9] are deduced by replacing the standard Brownian motion involved in the classical model with fractional Brownian motion. Further, Wang et al. [10,11] obtained a option pricing model with transaction costs in the fractional version of the Merton model. Gu et al. [12] deal with the option pricing with transaction costs by a fractional sub-diffusive Black-Scholes model. Liu et al. [13] proposed a pricing formula for the European option with transaction costs and provided an approximate solution of the nonlinear Hoggard-Whalley-Wilmott equation. Zhang et al. [14] solved the pricing problem of geometric average Asian option with transaction costs under fractional Brownian motion. Xiao et al. [15] used the sub-fractional Brownian to construct the warrants pricing model with transaction costs.

The above-mentioned models made great progress and coverd the gap of the classical Black-Scholes model, but these equations are still ones with integer-order derivatives. Comparing with these equations, the fractional differential equations provide an excellent instrument for description of memory and hereditary properties of various materials and processes. The introduction of fractional differential equation into the financial theory provides a new idea and tool for the researches of pricing theory. Wyss [16] presented the fractional Black-Scholes equation with a time-fractional derivative to price European call option. Cartea et al. [17] deduced the space-fractional diffusion models of option prices under three special processes of FMLS, CGMY and KoBoL in markets with jumps and priced barrier option FMLS model. Jumarie [18,19] derived the time and space-time fractional Black-Scholes equations and gave optimal fractional Merton's portfolio. Based on Jumarie' ideas, Liang et al. [20,21] gained a Black-Scholes model with time and space fractional derivatives. On the basis, Marom et al. [22], Xi

☆ The work is partially supported by National Natural Science Foundation of China (No. 71501031, No. 71171035, No. 71273044, No. 71571033, No. 11601068).

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http://dx.doi.org/10.1016/j.chaos.2017.05.043
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et al. [23], Song et al. [24] and Chen et al. [25] employed analytical and numerical methods to solve these fractional option pricing models. As far as we know, the study on the fractional derivative model with transaction costs is few. The aims of the work is to establish and solve the space-time fractional pricing model in the presence of transaction costs and testy the practicability of the results by the real data.

The paper has been organized as follows. In Section 2, the definition and properties of the modified Riemmna-Liouville derivative are introduced and the space-time fractional derivative model of option pricing is derived. In Section 3, the semi-analytical solutions of fractional model are solved by the enhanced technique and the finite difference method. In Section 4, fractional derivative model is tested by the data. Conclusions and discussions are presented in Section 5.

2. Fractional derivative model

In the *Section*, the space-time fractional Black-Scholes equation for European option pricing with transaction costs is derived under the definition of modified Riemmna-Liouville derivative.

2.1. Modified Riemmna–Liouville derivative

The modified fractional derivative is proposed by Jumarie to cover some shortages involved in the classical Riemann-Liouville derivative. Reviewing the literatures [18,19], the definition and main properties of the modified fractional derivative are described, as follows.

Definition 2.1 ([19] (Riemann–Liouville definition revisited)). (i) Assume that *f*(*x*) is a constant *K*. Then its fractional derivative of order α is

$$D_x^\alpha K = \begin{cases} \frac{K}{\Gamma(1-\alpha)} x^{-\alpha}, & \text{if } \alpha \le 0, \\ 0, & \text{if } \alpha > 0. \end{cases} \tag{1}$$

(ii) Assume that *f*(*x*) is not a constant. Then its fractional derivative of order α is

$$D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1}(f(\xi)-f(0))d\xi, & \text{if } \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & \text{if } 0<\alpha<1, \\ (f^{(\alpha-n)}(x))^{(n)}, & \text{if } n \le \alpha < n+1, n \ge 1. \end{cases} \tag{2}$$

Where Gamma function $\Gamma(z) = \int_0^\infty \tau^{z-1}\exp(-\tau)d\tau$. Especial for a positive integer *n*, $\Gamma(n) = (n-1)!$.

Under the modified Riemann–Liouville definition, Jumarie offered a generalized Taylor expansion on the single variable and multi-variable functions.

Proposition 2.1 ([19]). *Assume that the continuous function f*: $R \to R$, $x \to f(x)$ *has fractional derivative of order* $k\alpha$, *for any positive integer k and any* α, $0 < \alpha \le 1$, *then the following equality holds, which is*

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{(\alpha k)}}{\Gamma(1+k)} f^{(\alpha k)}(x), \quad 0 < \alpha \le 1, \tag{3}$$

where $f^{(\alpha k)}(\cdot)$ *is the derivative of order* αk *of f*(*x*).

Proposition 2.2 ([19]). *Multi-variable fractional Taylor's series*

$$f(x+\xi, y+\eta) = E_\alpha(\xi^\alpha D_x^\alpha) E_\alpha(\eta^\alpha D_y^\alpha) f(x,y). \tag{4}$$

Where E_α *is a Mittag-Leffler function and* $E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)}$.

In the approximation of order 2α, this series provides the equality

$$\begin{aligned} f(x+\xi, y+\eta) \cong{}& f(x,y) + \frac{1}{\Gamma(1+\alpha)}(f_x^{(\alpha)}(x,y)\xi^\alpha + f_y^{(\alpha)}(x,y)\eta^\alpha) \\ &+ \frac{1}{\Gamma(1+2\alpha)}(f_x^{(2\alpha)}(x,y)\xi^{2\alpha} + f_y^{(2\alpha)}(x,y)\eta^{2\alpha}) \\ &+ \frac{1}{(\Gamma(1+\alpha))^2} f_{xy}^{(2\alpha)}(x,y)\xi^\alpha\eta^\alpha. \end{aligned} \tag{5}$$

As a direct application of the fractional Taylors series, Jumarie gave the following corollaries.

Corollary 2.1 ([19]). *The following equalities hold, which are*

$$\begin{aligned} &D^\alpha x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-\alpha)x^{\gamma-\alpha}, \quad \gamma > 0, \\ &(u(x)v(x))^{(\alpha)} = u^{(\alpha)}(x)v(x) + u(x)v^{(\alpha)}(x), \\ &(f[(x)])^{(\alpha)} = f'_u u^{(\alpha)}(x) = f_u^\alpha(u)(u'(x))^{(\alpha)}. \end{aligned} \tag{6}$$

Corollary 2.2 ([19]). *Assume that f*(*x*) *and x*(*t*) *are two* $R \to R$ *functions which both have derivatives of order* α, $0 < \alpha \le 1$, *then one has the chain rule*

$$f_t^{(\alpha)}(x(t)) = \Gamma(2-\alpha)x^{\alpha-1} f_x^{(\alpha)}(x) x^{(\alpha)}(t). \tag{7}$$

2.2. Mathematical deduction

In the work, the replicating technique is adopted to establish fractional derivative model. The following assumptions are made in financial market with transaction costs.

I The change of the value for the replicating portfolio Π_t in $[t, t+dt]$ is subject to the fractional differential equation

$$d^H\Pi_t = X_1(t)(dS_t)^H + X_2(t)d^H D_t. \tag{8}$$

Where S_t and D_t denote the price of underlying asset and the riskless bond, respectively. $X_1(t) = X_1(t, S_t)$ and $X_2(t)$ are the corresponding shares. $H \in [0, 1]$ is Hurst exponent. When $1/2 < H \le 1$, the time sequence has long range dependence or long memory. When $H = 1/2$, the time sequence can be described by random walks. When $0 < H < 1/2$, the time sequence shows the antipermanence character.

The bond *D* is risk-less during the time *dt*, then it satisfies the following equality according to Refs.[20,21],

$$d^H D_t = rD_t(dt)^H. \tag{9}$$

Here, the form $(dt)^H$ comes from Refs. [18,19], where Jumarie extended the Maruyamas notation for Brownian motion $b(t, \alpha)$ and introduced $db(t,\alpha) = \sigma w(t)(dt)^\alpha$, and further gave the following Lemma.

Lemma 2.1 ([19]). *Let f*(*t*) *denote a continuous function, then the solution of* $dx = f(t)(dt)^\alpha$, $x(0) = x_0$ *is defined by the equality*

$$\int_0^t f(\tau)(d\tau)^\alpha = \alpha\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad 0 < \alpha \le 1. \tag{10}$$

Integration with respect to $(dt)^\alpha$ and its application can refer to Refs.[18,19].

II Transaction cost is a direct cost due to trading and it is the fixed proportion *c* of the trading amount for the underlying. It is expressed as

$$Cost = cS_t|\nu_t|, \tag{11}$$

where ν_t denotes the shares of the underlying that are bought ($\nu_t > 0$) or sold ($\nu_t < 0$) at the price S_t.

III Based on Refs.[20,21], the price S_t of the underlying asset follows the fractional exponential equation

$$(dS_t)^H = \mu S_t^H (dt)^H + \sigma S_t^H dB_H(t). \tag{12}$$

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