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A reliable numerical algorithm for the fractional vibration equation

Harendra Singh[a,*], H.M. Srivastava[b,c], Devendra Kumar[d]

[a] School of Mathematical Sciences, National Institute of Science Education and Research (NISER), Khurda 752050, Odisha, India
[b] Department of Mathematics and Statistics, University of Victoria, British Colombia V8W 3R4, Canada
[c] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan Republic of China
[d] Department of Mathematics, JECRC University, Jaipur 303905, Rajasthan, India

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ABSTRACT

The key purpose of this article is to introduce a numerical algorithm for the solution of the fractional vibration equation (FVE). The numerical algorithm is based on the applications of the operational matrices of the Legendre scaling functions. The main advantage of the numerical algorithm is that it reduces the FVE into Sylvester form of algebraic equations which significantly simplify the problem. Error as well as convergence analysis of the proposed scheme are shown. Numerical results are discussed taking different initial conditions and wave velocities involved in the problem. Numerical results obtained by using suggested numerical algorithm are compared with the existing analytical methods for the different cases of FVE.

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1. Introduction

It is necessary requirement to study the vibration of membranes because they play important role in various fields like analysing the two dimensional wave mechanics and propagation, bioengineering and used as a component of speakers, microphones and other devices. Vibration equation (VE) is used to describe the vibration of a membrane. The standard VE is given by

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, \; r \geq 0, \quad t \geq 0, \tag{1}$$

where component u is the displacement of particle at the position r at the time instant t,c is the wave velocity of free vibration,r is spatial domain and t is the time variable.

Some physical quantity depends on the past so fractional model of such differential equation becomes important to understand the physical model in better manner. Fractional calculus has many real applications in engineering and science like as biology [1,2], viscoelasticity [3–5], signal processing [6,7], control theory [8], chemistry [9], finance [10], etc. More details about the theory and applications of fractional differential equations can be found in [11].

In this paper, we will consider more general form of VE by replacing integer order time derivative by fractional order Liouville-Caputo derivative as follows:

$$\frac{1}{c^2}\frac{\partial^\alpha u(r,t)}{\partial t^\alpha} = \frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,t)}{\partial r}, \qquad 1 < \alpha \leq 2, \tag{2}$$

Subject to the initial conditions:

$$u(r,0) = g(r), \; \frac{\partial u(r,0)}{\partial t} = ch(r), \text{ for } 0 \leq r, t \leq 1, \tag{3}$$

There exist some analytical methods to solve fractional and integer order vibration equation such as homotopy perturbation method [12], modified decomposition method [13]. In [14], Das and Gupta proposed an analytical technique based on the homotopy analysis method for the solution of FVE. In [15], modified decomposition and variational iteration methods are used to solve this problem. Recently in an attempt Srivastava et al. [16] used the q-homotopy analysis transform technique and Laplace decomposition technique to obtain analytical solution of FVE. The operational matrix method (see [17–26]) is also used to solve linear and nonlinear problems pertaining to fractional calculus.

In the present paper, we are using a computational method which is based on the operational matrices of Legendre scaling functions. In this method, first we take finite dimensional approximation of unknown function. Further, using operational matrices in the FVE, we obtain a system of algebraic equations in Sylvester form whose solution gives approximate solution for the FVP. Error analysis of the proposed method is given. Convergence of approximate numerical solution to exact solution is shown in same sense as developed by J.F. Colombeau [27,28]. To show the applicability and accuracy of the proposed method we have compared the obtained results with existing analytical methods.

* Corresponding author.
E-mail addresses: harendrasingh@niser.ac.in, harendra059@gmail.com (H. Singh), harimsri@math.uvic.ca (H.M. Srivastava), devendra.maths@gmail.com (D. Kumar).

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2. Preliminaries

The fractional order differentiation and integration are defined as follows:

Definition 2.1. The Riemann–Liouville fractional order integral operator is given by

$$J^{\alpha}g(y)=\frac{1}{\Gamma(\alpha)}\int_0^y (y-t)^{\alpha-1}g(t)dt \quad \nu\alpha>0, y>0,$$

$$J^0 g(y)=g(y).$$

For the fractional Riemann–Liouville integration

$$J^{\alpha}y^k=\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}y^{k+\alpha}$$

Definition 2.2. The Liouville-Caputo fractional derivative of order β are defined as

$$D^{\beta}g(y)=J^{l-\beta}D^l g(y)=\frac{1}{\Gamma(l-\beta)}\int_0^y (y-t)^{l-\beta-1}\frac{d^l}{dt^l}g(t)dt,$$
$$l-1<\beta<l, y>0.$$

In present paper we use the following property of fractional Liouville-Caputo derivative,

$$D^{\beta}A=0 \quad (\text{A is a constant}),$$

$$D^{\beta}y^k=\begin{cases}\dfrac{\Gamma(k+1)}{\Gamma(k+1-\beta)}y^{k-\beta}, & \text{for } k\in \mathrm{N}_0 \text{ and } k\geq\lceil\beta\rceil,\\ 0, & \text{for } k\in \mathrm{N}_0 \text{ and } k<\lceil\beta\rceil,\end{cases}$$

where symbols have their usual meanings, while $\mathrm{N}=\{1,2,3,......\}$ and

$\mathrm{N}_0=\{0,1,2,......\}$.

Lemma 2.1. [12]: *If $p-1<\alpha\leq p$, $p\in N$, and $h\in L^2[0,1]$ then $D^{\alpha}I^{\alpha}h(y)=h(y)$ and*

$$I^{\alpha}D^{\alpha}h(y)=h(y)-\sum_{k=0}^{p-1}h^{(k)}(0^+)\frac{y^k}{k!}, \quad y>0.$$

The Legendre scaling functions $\{\phi_i(t)\}$ in one dimension are defined by

$$\phi_i(t)=\begin{cases}\sqrt{(2i+1)}P_i(2t-1), & \text{for } 0\leq t<1.\\ 0, & otherwise,\end{cases}$$

where $P_i(t)$ is Legendre polynomials of order i on the interval $[-1,1]$, given explicitly by the following formula;

$$P_i(t)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{t^k}{(k!)^2} \quad . \tag{4}$$

Using one dimensional Legendre scaling functions, we construct two dimensional Legendre scaling function ϕ_{i_1,i_2},

$$\phi_{i_1,i_2}(x,t)=\phi_{i_1}(x)\phi_{i_2}(t),\ i_1,i_2\in N_0.$$

From the above formula it is clear that two dimensional Legendre scaling functions are orthogonal;

$$\int_0^1\int_0^1 \phi_{i_1,i_2}(x,t)\phi_{j_1,j_2}(x,t)dxdt=\begin{cases}1, & i_1=j_1 \text{ and } i_2=j_2,\\ 0, & otherwise.\end{cases}$$

and (ϕ_{i_1,i_2}) form a complete orthonormal basis.

So, a function $h(x,t)\in L^2([0,1]\times[0,1])$, can be approximated as

$$h(x,t)\cong\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}c_{i_1,i_2}\phi_{i_1,i_2}(x,t)=C^T\phi(x,t), \tag{5}$$

where $C=[c_{0,0},....,c_{0,m_2},...,c_{m_1,1},...,c_{m_1,m_2}]^T$;

$\phi(x,t)=[\phi_{0,0}(x,t),....,\phi_{0,m_2}(x,t),...,\phi_{m_1,1}(x,t),...,\phi_{m_1,m_2}(x,t)]^T$.

The coefficients c_{i_1,i_2} in the Fourier expansions of $h(x,t)$ are given by the formula,

$$c_{i_1,i_2}=\int_0^1\int_0^1 h(x,t)\phi_{i_1,i_2}(x,t)dxdt. \tag{6}$$

Using matrix notation Eq. (5) can be written as,

$$h(x,t)\cong\phi^T(x)C\phi(t), \tag{7}$$

where $\phi(x)=[\phi_0(x),........,\phi_{m1}(x)]^T$, $\phi(t)=[\phi_0(t),........,\phi_{m2}(t)]^T$ and $C=(c_{i_1,i_2})_{(m_1+1)\times(m_2+1)}$.

3. Operational matrices

Theorem 3.1. *Let $\phi(x)=[\phi_0(x),\phi_1(x),........,\phi_m(x)]^T$, be Legendre scaling vector and consider $\alpha>0$, then*

$$J^{\alpha}\phi_i(x)=J^{(\alpha)}\phi(x), \tag{8}$$

where $J^{(\alpha)}=(\sigma(i,j))$, is $(m+1)\times(m+1)$ operational matrix of fractional integral of order α and its (i,j) th entry is given by

$$\sigma(i,j)=(2i+1)^{1/2}(2j+1)^{1/2}\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{i+j+k+l}$$
$$\times\frac{(i+k)!(j+l)!}{(i-k)!(j-l)!(k)!(l!)^2(\alpha+k+l+1)\Gamma(\alpha+k+1)}$$
$$\times 0\leq i,j\leq m.$$

Proof. Please see [29].

Theorem 3.2. *Let $\phi(x)=[\phi_0(x),\phi_1(x),........,\phi_m(x)]^T$, be Legendre scaling vector and consider $\beta>0$, then*

$$D^{\beta}\phi_i(x)=D^{(\beta)}\phi(x), \tag{9}$$

where $D^{(\beta)}=(\lambda(i,j))$, is $(m+1)\times(m+1)$ operational matrix of Liouville-Caputo fractional derivative of order β and its (i,j) th entry is given by

$$\lambda(i,j)=(2i+1)^{1/2}(2j+1)^{1/2}\sum_{k=\lceil\beta\rceil}^{i}\sum_{l=0}^{j}(-1)^{i+j+k+l}$$
$$\times\frac{(i+k)!(j+l)!}{(i-k)!(j-l)!(k)!(l!)^2(k+l+1-\beta)\Gamma(k+1-\beta)}.$$

Proof. Please see [30]. □

Theorem 3.3. *Let $\phi(x)=[\phi_0(x),\phi_1(x),........,\phi_m(x)]^T$, be Legendre scaling vector, then*

$$x\phi_i(x)=E\phi(x), \tag{10}$$

where $E=(\mu(i,j))$, is $(m+1)\times(m+1)$ operational matrix and its (i,j) th entry is given by

$$\mu(i,j)=(2i+1)^{1/2}(2j+1)^{1/2}\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{i+j+k+l}$$
$$\times\frac{(i+k)!(j+l)!}{(i-k)!(j-l)!(k!)^2(l!)^2(k+l+2)}$$
$$\times 0\leq i,j\leq m.$$

Proof. Using the Legendre scaling function of degree i, we get

$$x\phi_i(x)=(2i+1)^{1/2}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{1}{(k!)^2}x^{k+1}$$

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