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On the complexity of economic dynamics: An approach through topological entropy

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ABSTRACT

In this paper we compute topological entropy with prescribed accuracy for different economic models, showing the existence of a topologically chaotic regime for them. In order to make the paper self-contained, a general overview on the topological entropy of continuous interval maps is given. More precisely, we focus on piecewise monotone maps which often appear as dynamical models in economy, but also in population growth and physics. Our main aim is to show that when topological entropy can be approximated up to a given error, it is a useful tool which helps to analyze the chaotic dynamics in one dimensional models.

1. Introduction

Chaos is a very popular notion of study for both theoretical and practical scientists and many models from natural and social sciences have been analyzed showing the existence of so–called chaotic behavior. In most cases, this chaotic behavior is shown by using numerical simulations made by computer without any error control. Nevertheless, these models and the simulations made to understand their dynamics, offer a wide field of study for mathematicians interested in giving more accurate and consistent approaches to model dynamics, which is not generally easy.

In the conference “100 years after Poincaré” held in Gijón (Spain),[1] J. A. Yorke, one of the fathers of chaos theory gave a conference entitled “the many facets of chaos”. He argued that we should study chaos from different points of view using different techniques: topological, statistical and others, to have a clearer vision of chaos. For instance, Li and Yorke chaos [27] is a topological notion which sometimes is not physically observable. A well-known example of this is a smooth enough unimodal interval map with positive entropy[2] and negative Schwarzian derivative that has a periodic orbit as its attractor. It is clear that such examples make chaos theory both rich and often “confusing”, especially for biologists and economists.

In this paper, we highlight the fact that topological entropy can be computed with prescribed accuracy to analyze the chaotic behavior of dynamical systems given by some economic models, going further than the numerical estimations that are usually taken and considered as true, even when error analysis is not carried out. So the reliability of simulations is omitted in most cases simply assuming the veracity of the results. However, its study deserves more attention, see for example [28]. We must point out that models usually depend on parameters and our main aim is to show that topological entropy can be a useful tool to provide confident parameter regions where chaos is possible. Although orbits are used to compute topological entropy, the orbit length is at most 2000 points and so, rounded off effects should not have a significative influence on the final result.

Topological complexity is not always physically observable. The reason comes from the fact that complex behavior can exist but only in a set of Lebesgue measure equal to zero, which means that numerical simulations show a simple behavior with probability equal to one. In this paper, “complex behavior” means “topological complex behavior”, which is different from the statistical properties of the dynamical system. The estimations that we make with prescribed accuracy are different from rigorous computations with prescribed accuracy in which numerical errors and rounding are considered. Thus, we have to distinguish between topological complexity, physically observable complexity and observed complexity when we make numerical simulations, see [21]. Nevertheless, the real possibilities of estimating topological entropy versus other notions such as Kolmogorov–Sinai entropy with greater constraints, [22], makes it a source of valuable information for studying dynamical systems.

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[1] http://www.unioviedo.es/ds100Poincare.

[2] Positive topological entropy is often taken as a definition of chaos and among others implies the existence of Li and Yorke chaos [7].

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Let us point out that there are rigorous numerical methods which are free from orbit computation (see [21,22]), which, especially in the case of chaotic systems, can affect the results when the orbit length is big enough. Clearly, these “orbit-free numerical methods”, which allow the computation of statistical properties of dynamical systems as invariant measures or metric entropy, are more rigorous than those that use orbits. In our opinion, developing effective numerical methods that allow us to compute the topological entropy approximately without computing the orbits will be very interesting.

The paper is organized as follows. In Section 2 we introduce the notion of topological entropy and its relationship to chaos. Section 3 is devoted to the computation of the topological entropy in a theoretical way. We emphasize results related with piecewise monotone maps which appear in a natural way in economic models. In Section 4, we show numerical methods used to compute the topological entropy with a prescribed accuracy. Finally, in Section 5, some applications to economic dynamics are shown. In particular, we present some models, depending on one or several parameters, for which topological entropy is useful in order to decide the parametric region where chaos can be found.

2. Topological entropy and chaos

The mathematical notion of entropy has its roots in physics. In thermodynamics, entropy is a measure of the number of ways in which a thermodynamic system may be arranged, often taken as a measure of disorder [18]. However, Shannon [39] introduced entropy in the frame of information theory. In these terms, *“entropy is a statistical parameter which measures in certain sense, how much information is produced on the average for each letter of a text in the language. If the language is translated into binary digits (0 or 1) in the most efficient way, the entropy H is the average number of binary digits required per letter of the original language”*, [39]. Shannon's entropy quantifies the expected value of the information contained in a message.

Shannon's notion of entropy was adapted in the setting of discrete dynamical systems. First, Kolmogorov and Sinai [25,40] introduced the metric or measure-theoretic entropy as follows. Let X be endowed with a probability measure μ on a σ-algebra of X and let T: $X \to X$ be a measure preserving transformation, that is, $\mu(A) = \mu(T^{-1}A)$ for any measurable set $A \subseteq X$. The entropy of T with respect to a measurable finite partition $\xi = \{C_1, \dots, C_k\}$ is defined by

$$h_\mu(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_n) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu(\xi_n),$$

where $\xi_n = \vee_{i=0}^{n-1} T^{-i}\xi$ is the partition of X by the sets $C_{i_1} \cap T^{-1}(C_{i_2}) \cap \dots \cap T^{-(n-1)}(C_{i_n})$, $C_{i_1}, C_{i_2}, \dots, C_{i_n} \in \xi_n$ and $H_\mu(\xi_n) = -\sum_{C\in\xi_n} \mu(C)\log(\mu(C))$. Then, metric entropy is defined as

$$h_\mu(T) = \sup_\xi h_\mu(T,\xi).$$

The underlying idea of metric entropy is to measure the speed with which the transformation T cuts smaller and smaller pieces under iteration.

Related to the previous definitions of metric entropy we find an approximation for the notion of topological entropy. In fact, for continuous transformations on compact metric space, the so-called variational principle for topological entropy states that the supremum of the metric entropies over all invariant probability measures of X is equal to the topological entropy, [42, Chapter 8]. Let us introduce the notion of topological entropy by means of topological objects.

Unless otherwise stated, throughout the paper we assume a difference equation $x_n = f(x_{n-1})$, where f: $X \to X$ is a continuous map on a compact metric space X. We can also speak of a discrete dynamical system (X, f), which is the natural frame where topological entropy makes sense. So, the *topological entropy* is a non–negative number $h(f)$ which measures the complexity of the map f, or the difference equation, or the discrete dynamical system (X, f). In fact, topological entropy measures the exponential growth rate of the number of orbits that can be distinguished as time increases. Two equivalent definitions of topological entropy have been given. One of them by Adler, Konheim and McAndrew (1965) [1] (using open covers) and the other by Bowen [11] (via separated and spanning sets).

First, we introduce the notion by using open covers, which is similar some ways to metric entropy. For this, we do not need any metric on X, which has to be compact and Hausdorff. Let f: $X \to X$ be a continuous map and let $\mathcal{A}$ be a finite open cover of X. Then $f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}$ is an open cover. If $\mathcal{B}$ is another open cover of X, let $\mathcal{A} \vee \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$ be the joining of $\mathcal{A}$ and $\mathcal{B}$. The minimal cardinality of any subcover of X chosen from $\mathcal{A}$ is denoted by $\mathcal{N}(\mathcal{A})$. Now the definition by Adler, Konheim and McAndrew of topological entropy is as follows.

Definition 1 (Adler, Konheim and McAndrew, [1]). Let f: $X \to X$ be a continuous map of a Hausdorff compact space X. Let $\mathcal{A}$ be a finite open cover of X.

1. The topological entropy of f on $\mathcal{A}$ is

$$h(f,\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{A})\right).$$

2. The topological entropy of f is $h(f) = \sup_{\mathcal{A}} h(f,\mathcal{A})$.

Topological entropy is invariant under conjugation, that is if f: $X \to X$ and g: $Y \to Y$ are maps such that there is a homeomorphism ϕ: $X \to Y$ with $\phi \circ f = g \circ \phi$, then $h(f) = h(g)$, see [2, Chapter 4].

Now, we introduce Bowen's notion with separated sets, skipping the equivalent definition for spanning sets that can be read in [11]. Here a metric on X is needed and, as we have already stated, Bowen's and open cover's definitions of topological entropy are equivalent when X is metric and compact.

Definition 2 (Bowen, [11]). Let f: $X \to X$ be a continuous map of a compact metric space (X, d). For $\epsilon > 0$ and $n \in \mathbb{N}$, we say that the subset $E \subset X$ is a (n, ϵ)-separated set if for every $x, y \in E$, $x \neq y$, there exists k, $0 \le k < n$, such that $d(f^k(x), f^k(y)) > \epsilon$. Then, the topological entropy of f, denoted by $h(f)$ is defined to be

$$h(f) = \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty} \frac{1}{n} \log N(n,\epsilon)\right),$$

where $N(n, \epsilon)$ is the maximum cardinality of a (n, ϵ)-separated set.

Thus, topological entropy measures the number of different orbits of the system. Suppose we do not distinguish between points that are less than ϵ apart. Then $N(n, \epsilon)$ represents the number of *distinguishable orbits* of length n, and if this number grows like approximately e^{nh}, then h is the topological entropy.

Observe that in the definition of topological entropy only the metric d and the induced topology are involved in counting the number of orbits, while metric entropy takes its value from a set of full measure, being “blind” for zero measure sets. This notion is different from the notion of KS entropy, that measures the exponential growth rate of *statistical relevant* distinguishable orbits respect to an invariant measure. A measure is also involved in the definition of Shannon entropy. In any case, a deeper look at the definitions is not necessary to see that computing topological entropy from the definitions is not generally easy. The question of finding methods to compute topological entropy was raised by recognized mathematicians such as John Milnor [32].