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Weak measure expansiveness for partially hyperbolic diffeomorphisms

Manseob Lee

Department of Mathematics, Mokwon University, Daejeon, 302-729, Republic of Korea

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ABSTRACT

Let M be a closed connected smooth Riemannian manifold with $\dim M \geq 2$ and let $f: M \to M$ be a diffeomorphism. In the paper, we show that C^1 generically, if a diffeomorphism f does not present a homoclinic tangency then it is weak Lebesgue measure expansive and, as an example, we find a partially hyperbolic diffeomorphism which is not weak measure expansive. Moreover, for a surface, if a diffeomorphism f has a homoclinic tangency then there is a diffeomorphism g C^1 close to f such that g is not weak measure expansive.

1. Introduction

Expansivity is very closely related to hyperbolic structure of the dynamical systems. In fact, Mañé [17] proved that if a diffeomorphism belongs to the C^1 interior of the set of every expansive diffeomorphisms then it is quasi-Anosov. Let M be a closed connected smooth Riemannian manifold with $\dim M \geq 2$, and let Diff(M) be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in$ Diff(M). We say that f is *expansive* if there is $\delta > 0$ such that for $x, y \in M$ if $d(f^i(x), f^i(y)) < \delta$ for all $i \in \mathbb{Z}$ then $x = y$. From Utz [22] who introduced the notion of expansiveness firstly, various notions of expansiveness were introduced such as continuum-wise expansive [11], N-expansive [15], measure expansive [14], weak measure expansive [2], entropy expansive [6], etc.

Let Λ be a closed f-invariant set. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then f is Anosov. It is known that if Λ is hyperbolic then it is expansive. For the probabilistic view point, Morales and Sirvent [14] introduced a general type of expansiveness which is called measure expansive. For any $x \in M$ and any $\delta > 0$, we define $\Gamma_\delta(x, f) = \{y \in M : d(f^i(x), f^i(y)) \leq \delta$ for all $i \in \mathbb{Z}\}$, called a *dynamic δ-ball* of f. It is clear that if f is expansive then $\Gamma_\delta(x, f) = \{x\}$ for all $x \in M$. Let μ be a Borel probability measure of M. Denote by $\mathcal{M}(M)$ the set of Borel probability measures on M endowed with weak* topology. We say that $\mu \in \mathcal{M}(M)$ is *atomic* if there is a point $x \in M$ such that $\mu(\{x\}) \neq 0$. Denote by $\mathcal{M}^*(M)$ the set of all nonatomic $\mu \in \mathcal{M}(M)$ and $\mathcal{M}^*_f(M)$ the set of all nonatomic invariant $\mu \in \mathcal{M}(M)$. It is known that $\mathcal{M}(M)$ is compact. We say that f is *μ expansive* if there is $\delta > 0$ such that $\mu(\Gamma_\delta(x, f)) = 0$ for all $x \in M$, where $\delta > 0$ is called an μ expansive constant of f. We say that f is *measure expansive* if there is $\delta > 0$ such that f is μ expansive for all $\mu \in \mathcal{M}^*(M)$. It is clear that if f is expansive then it is measure expansive, but the converse is not true (see [3]).

Ahn et al. [2] introduced weak measure expansiveness which is a more general notion of measure expansiveness. Let $P = \{A_i \subset M : i = 1, 2, \ldots, n\}$ be a finite collection of subsets M. For any $\delta > 0$, we say that $P = \{A_i \subset M : i = 1, 2, \ldots, n\}$ is a *finite δ partition* of M if (i) for each $i = 1, 2, \ldots, n$, A_i is measurable, $\mathrm{int}(A_i) \neq \emptyset$ and $\mathrm{diam} A_i \leq \delta$. (ii) $A_i \cap A_j = \emptyset$ and $\bigcup_{i=1}^n A_i = M$.

Since M is compact, for any $\delta > 0$ we can make a finite δ partition $P = \{A_i : i = 1, \ldots, n\}$ of M such that $\mathrm{diam} A_i \leq \delta$ for all $i = 1, 2, \ldots, n$. For any $x \in M$, we define $\Gamma_P(x, f) = \{y \in M : f^i(y) \in P(f^i(x))$ for all $i \in \mathbb{Z}\}$. The set $\Gamma_P(x, f)$ is called the *dynamic P-ball* of f centered at $x \in M$ and $P(x)$ denotes the element of P containing x. For any $\mu \in \mathcal{M}^*(M)$, we say that f is weak μ expansive if there is $\delta > 0$ and a finite δ-partition $P = \{A_i : i = 1, 2, \ldots, n\}$ of M

E-mail addresses: lmsds@mokwon.ac.kr, lmspwr@gmail.com

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such that $\mu(\Gamma_P(x, f)) = 0$ for all $x \in M$, where $\delta > 0$ is an weak measure expansive constant of f.

Definition 1.1. We say that f is weak measure expansive if f is weak μ expansive for all $\mu \in \mathcal{M}^*(M)$.

In [2, Example 2.4], they showed that a homeomorphism f is weak measure expansive but it is not measure expansive. For convenience, we introduce the example. for transpose dot in lemma

Example 1.2 [2], Example 2.4]. Let f: $S^1 \to S^1$ be an irrational rotation map. Then f is weak measure expansive. But f is not m expansive, where $m \in \mathcal{M}^*(S^1)$ is the Lebesgue measure on S^1.

Proof. Let $S^1 = [0, 2\pi)$. For any small $0 < \delta < \delta/2$, let $P = \{A_i \subset [0, 2\pi) : i = 1, \cdots, n\}$ be a finite δ-partition of $[0, 2\pi)$ such that each A_i is a half-open interval with diam$A_i < \delta$. For any $x \in S^1$, let $x \in A_j$ for some $j \in \{1, 2, \ldots, n\}$. We show that $\Gamma_P(x, f) = \{x\}$. For this, take $y \in A_j$ with $x \neq y$ such that $x < y$. Put $d(x, y) = \epsilon > 0$. Since every rotation map is an isometry, we know $d(x, y) = d(f^i(x), f^i(y))$ for all $i \in \mathbb{Z}$. Take an end point $z \in A_k$ for some $k \in \{1, 2, \ldots, n\}$ and open ball $B_{\epsilon/2}(z)$ containing z. Since every orbit of f is dense, there is $l \in \mathbb{Z}$ such that $f^l(x) \in A_{k-1} \cap B_{\epsilon/2}(z)$. Since f is an orientation preserving map and an isometry, $f^l(y)$ must be an element of A_k. This means $\Gamma_P(x, f) = \{x\}$ and so $\mu(\Gamma_P(x, f)) = 0$ for all $\mu \in \mathcal{M}^*(S^1)$. Thus f is weak measure expansive.

On the other hand, let $m \in \mathcal{M}^*(S^1)$ be the Lebesgue measure on S^1. Since f is an isometry, we know that for every $x \in S^1$

$$\Gamma_\delta(x) = \{y \in S^1 : d(f^i(x), f^i(y)) \leq \delta \text{ for } i \in \mathbb{Z}\} = B_\delta[x],$$

where $B_\delta[x]$ is the closed δ-ball centered at x. Then we have

$$m(\Gamma_\delta(x)) = m(B_\delta[x]) > 0.$$

This means that f is not m-expansive. □

A diffeomorphism f exhibits a *homoclinic tangency* if there is a hyperbolic periodic point p whose invariant manifolds $W^s(p)$ and $W^u(p)$ have a non-transverse intersection. The set of C^1 diffeomorphisms that have a homoclinic tangency will be denoted $\mathcal{HT}$. [13] proved that every diffeomorphism $f \in \text{Diff}(M) \setminus \overline{\mathcal{HT}}$ is entropy expansive.

Weak measure expansiveness is not equal to entropy expansiveness. It is well known that the identity map is entropy expansive, but it is not weak measure expansive which was proved by [2, Lemma 2.5]. Indeed, let X be a compact connected metric space and f: $X \to X$ be the identity map. For any $\delta > 0$, let $P = \{A_1, A_2, \ldots, A_n\}$ be a finite δ-partition of X. Then $\Gamma_P(x, f) = P(x) = A_i$ for some $i \in \{1, \ldots, n\}$. Choose $A_i \in P$ such that $\mu(A_i) > 0$. This means $\mu(\Gamma_P(x, f)) > 0$ for all $x \in A_i$. Thus if f: $X \to X$ is the identity map then it is not weak measure expansive.

We say that a f-invariant closed set Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. The set Λ is *partially hyperbolic* if there is a dominated splitting $E \oplus F$ of $T_\Lambda M$ such that either E is contracting or F is expanding. We say that a compact f-invariant set $\Lambda \subset M$ is *strongly partially hyperbolic* if the tangent bundle $T_\Lambda M$ has a dominated splitting $E^s \oplus E^c \oplus E^u$ and there exist $C > 0$ and $0 < \lambda < 1$ such that

(a) for all $v \in E^s$, $u \in E^c$, we have

$$\|D_x f^n(v)\| \cdot \|D_{f^n(x)} f^{-n}(u)\| \leq C\lambda^n \|v\| \cdot \|u\|,$$

for all $x \in \Lambda$, $n \geq 0$, and

(b) for all $w \in E^u$, $u \in E^c$, we have

$$\|D_{f^n(x)} f^{-n}(w)\| \cdot \|D_x f^n(v)\| \leq C\lambda^n \|w\| \cdot \|u\|$$

for all $x \in \Lambda$, $n \geq 0$,

Note that if Λ is hyperbolic for f then it is strongly partially hyperbolic and E^c is empty. A subset $\mathcal{G} \subset \text{Diff}(M)$ is *residual* if it contains a countable intersection of open and dense subsets of Diff(M). A dynamic property is C^1*generic* if it holds in a residual subset of Diff(M). Note that there is a residual set $\mathcal{G} \subset \text{Diff}(M)$ such that for any $f \in \mathcal{G} \cap \overline{\mathcal{HT}}$, f is not entropy expansive (see [13, Remark 1.1]). Recently, Pacifico and Vieitez [19] proved that there is a residual subset $\mathcal{G}$ of $\text{Diff}(M) \setminus \overline{\mathcal{HT}}$ such that for any Borel probability measure μ which is absolutely continuous with respect to Lebesgue, $f \in \mathcal{G}$ is μ-expansive. From these facts, we have the following.

Theorem A. *There is a $\mathcal{G}$ residual subset of $Diff(M) \setminus \overline{\mathcal{HT}}$ such that for any Borel probability measure μ which is absolutely continuous with respect to Lebesgue, $f \in \mathcal{G}$ is weak μ-expansive.*

Now, we consider following problem: *If a diffeomorphism f is partially hyperbolic which is far away from homoclinic tangencies then is it weak measure expansive?*

Let $M = \mathbb{T}^3$ and let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a diffeomorphism. In [16, Theorem B], Mañé constructed a robustly nonhyperbolic transitive diffeomorphism $f \in \text{Diff}(\mathbb{T}^3)$. By [10, Theorem B], every robustly transitive diffeomorphism f on $\mathbb{T}^3$ is partially hyperbolic. Thus we can find a partially hyperbolic diffeomorphism f on $\mathbb{T}^3$ such that f is robustly nonhyperbolic transitive. Denoted by $\mathcal{RT}(\mathbb{T}^3)$ the set of all robustly transitive diffeomorphism on $\mathbb{T}^3$.

Theorem B. *Let $f \in \mathcal{RT}(\mathbb{T}^3)$. Then there is g C^1 close to f such that g is not weak measure expansive.*

2. Proof of Theorems

2.1. Proof of Theorem A

For a closed f-invariant set Λ, it admits a dominated splitting. Then if the dominated splitting can be written as the following

$$T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_i \oplus E_{i+1} \oplus \cdots \oplus E_k,$$

then we say that the sum is *dominated* if for all i the sum

$$(E_1 \oplus E_2 \oplus \cdots \oplus E_i) \oplus (E_{i+1} \oplus E_{i+2} \oplus \cdots E_k)$$

is dominated. Note that the decomposition is called the *finest* dominated splitting if we can't decompose in a non-trivial way subbundle E_i appearing in the splitting. In the above, if $E_1 = E^s$ and $E_k = E^u$ then Λ is partially hyperbolic. For a partially hyperbolic diffeomorphism, Burns and Wilkinson [7] showed the following lemma (see [19, Proposition 3.2]).

Lemma 2.1. *Let Λ be a compact f-invariant set with a partially hyperbolic splitting,*

$$T_\Lambda M = E^s \oplus E_1^c \oplus \cdots \oplus E_k^c \oplus E^u.$$

Let $E^{cs,i} = E^s \oplus E_1^c \oplus \cdots \oplus E_i^c$ and $E^{cu,i} = E_i^c \oplus \cdots \oplus E_k^c \oplus E^u$ and consider their extensions $\widetilde{E}^{cs,i}$ and $\widetilde{E}^{cu,i}$ to a small neighborhood of Λ. Then for any $\epsilon > 0$ there exist constants $R > r > r_1 > 0$ such that for any $x \in \Lambda$, the neighborhood $B(x, r)$ is foliated by foliations $\widehat{W}^u(x), \widehat{W}^s(x), \widehat{W}^{cs,i}(x)$ and $\widehat{W}^{cu,i}(x) (i = 1, \ldots, k)$ such that for each $\sigma \in \{u, s, (cs, i), (cu, i)\}$ the following properties hold.

(a) *Almost tangency of invariant distributions. For each $y \in B(x, r)$, the leaf $\widehat{W}_x^\sigma(y)$ is C^1, and the tangent space $T_y\widehat{W}_x^\sigma(y)$ lies in a cone of radius ϵ about $\widetilde{E}^\sigma(y)$.*

(b) *Coherence. $\widehat{W}_x^s$ subfoliates $\widehat{W}_x^{cs,i}$ and $\widehat{W}_x^u$ subfoliates $\widehat{W}_x^{cu,i}$ for each $i \in \{1, \ldots, k\}$.*

(c) *Local invariance.* For each $y \in B(x, r)$ *we have $f(\widehat{W}_x^\sigma(y, r_1)) \subset \widehat{W}_{f(x)}^\sigma(f(y))$ and $f^{-1}(\widehat{W}_x^\sigma(y, r_1)) \subset \widehat{W}_{f^{-1}(x)}^\sigma(f^{-1}(y))$, where $\widehat{W}_x^\sigma(y, r_1)$ is the connected components of $\widehat{W}_x^\sigma(y) \cap B(y, r_1)$ containing y.*