Chaos, Solitons and Fractals 103 (2017) 364–370

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

# A simple and fast method for valuing American knock-out options with rebates

Kyunghyun Park, Junkee Jeon*

*Department of Mathematical Sciences, Seoul National University, Seoul 08826, Republic of Korea*

ARTICLE INFO

*Article history:*
Received 26 February 2017
Revised 7 June 2017
Accepted 23 June 2017

*Keywords:*
American knock-out options
Free boundary problems
Laplace–Carson transforms
Numerical Laplace inversion
Rebate

ABSTRACT

In this paper, we derive an analytic formula for the American knock-out options with rebate. Rather than using a probabilistic method, we use the Laplace–Carson Transform(LCT) method to induce a simple functional equation associated with the complex problem of option pricing Partial Differential equation with free boundary. The transformed value of free boundary could be solved by applying Newton's method. Lastly, numerical Laplace inversion techniques are used to solve for the wanted free boundary value and the options value.



## 1. Introduction

American option is an option, which has the characteristics of exercising their right before the option holder expires. Due to the characteristics of the possibility of early exercise of the option, American options have shown a vast number of trade volumes. So its pricing research still continues till today. So far, the purpose of American option pricing was to solve for the early exercise boundary of an option through a pricing mechanism. These days, banks, corporations, and institutional investors use non-standard or exotic options for their management of risk. There is no doubt that standard American option is also frequently used as a risk management tool, however, this option has its difficulty when applied in a hedging situation under certain situations. On top of this, the use of the standard option could be 'over' hedging for certain corporations or institutions and could lead to inefficiency of investing in a higher cost. As a result, non-standard option is not only efficient in risk hedging but also has the effect of lowering the hedging cost.

This paper focuses on option pricing of the barrier options down and out with rebate (knock-out) options, which is one of the various exotic options. Barrier option is an option on an underlying asset whose existence depends on the underlying assets price reaching pre-set barrier level. In barrier option, there exists 4 main types and among them, the down and in option is an option that becomes activated only if the price of the underlying asset falls below a pre-determined barrier price level of maturity. Also, the opposite of a down and in option is a down-and out option that becomes null and void if the price falls below a certain barrier price. These options are path dependent options often dealt with in financial markets. However, barrier option holders always hold the (knocked-in, knocked-out) risk of the option in case of losing its value due to the barrier. To compensate for the underlining risk, barrier option has offered rebate in case of a worst-case scenario in markets, which is used in many different situations and trades. This rebate could provide a certain compensating prices or could change according to the time, meaning, could be time dependent rebate, which relies on the time until expiration. In this paper, we offered a pricing of an option in a down and out situation that offers a time dependent rebate.

In 1973, Merton [1] first attempted inducing a price formula about a general barrier option. Since then, Rich [2]and Wong and Kwok [3] published papers attempting to induce a price formula on one-asset barrier option and multi-asset barrier option. Generally, there are many analytic techniques using the classic Black–Scholes–Mertons formula or techniques of pricing that are numerical but PDE itself limits us to induce a solution because of its complexity and difficulty. Therefore, in effect, there has been many researches transforming PDE to a simple algebraic equation in frequency domain. Recently Le et al. suggested a method of down and out call option [4] and up and out put option [5], respectively, with an existing rebate, which is induced as a Fourier-Sine transform. Unfortunately, this method also has a disadvantage of the fact that the option value has a quite complex integral form with singular-

* Corresponding author.
*E-mail addresses:* luminary9699@snu.ac.kr (K. Park), hollman@snu.ac.kr, junkeejeon@gmail.com (J. Jeon).

http://dx.doi.org/10.1016/j.chaos.2017.06.026
0960-0779/

ity, not to mention the time costing in the numerical calculation process.

In this paper, the core method used in pricing is applying the Laplace–Carson transform(LCT). Unlike the aforementioned analytic approach, not only does LCT have a merit because the converted transform from PDE can be simplified. The induced solution after the conversion can be converted again through a simple algorithm of the Laplace inversion transform without it being time consuming. Above all, there has been plenty of papers proving the accuracy of induced solution using the Laplace-Carson transform. In a paper written by Kimura [6], LCT was used in pricing of finite-lived Russian options and Wong and Zhao [7] also used this method in valuing American option as a CEV model. Not to mention that in a paper by Kang et al. [8], Kang et al.used American strangle option which is one of the barrier option in pricing. This paper presents the two options American down and in, down and out, and the method of inducing a simple algebraic equation in the two options pricing in PDE.

This paper is constructed in the following order. In Section 2, we formulates the free problem of American knock-out options with rebate using the standard variational inequality technique. In Section 3, we obtain the general solution for the value of American knock-out options with rebate in frequency domain and algebraic equation whose solution is the value of early exercise free boundaries in frequency domain. In Section 4, we finally derive the free boundaries and the value of American knock-out options with rebate using numerical Laplace inversion scheme to the result of Section 4. We also present some numerical solutions and plots of the value of American knock-out options with rebate.

## 2. Model formulation: free boundary problems

In this paper, we consider the valuation of American knock-out options with rebate under the setting of Black–Scholes. Then we assume that the risk-neutral process of the underlying asset price evolves according to the stochastic differential equation(SDE):

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t \tag{2.1}$$

where $r > 0$ represent the risk-free interest rate, $q > 0$ is the dividend yield, and $\sigma > 0$ is the constant volatility of $S_t$, and $W_t$ is a standard Brownian motion on filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq}, \mathbb{P})$, where $(\mathcal{F}_{t\geq 0}) \equiv \mathbb{F}$ is the natural filtration generated by $(W_t)_{t\,\geq\,0}$. Now, we mathematically formulate about two types of American knock-out options with rebate, i.e., American down-and-out call option and American up-and-out put option.

### 2.1. American down and out call option with rebates

Let $C_{do}(t, S)$ be the American down and out call option with rebates price and the down barrier $D$ satisfying $D < S_0$ and $D < K$. Let us define $\tau_D$ as the stopping time of $\mathcal{F}_t$ with $\tau_D = \inf\{t > 0 \mid S_t = D\}$. Then, in the absence of arbitrage opportunities, the value $C_{do}(t, S)$ is a solution of the optimal stopping problem

$$C_{do}(t,S) = \max_{t\leq\xi_t\leq T} \mathbb{E}\Big[e^{-r(\xi_t - t)}(S_{\xi_t}-K)^+\mathbf{1}_{\{\tau_D\geq T\}} + e^{-r(\xi_t - t)}R(\xi_t)\mathbf{1}_{\{\tau_D<T\}} \mid S_t = S\Big] \tag{2.2}$$

where $\mathbb{E}$ is the expectation under the risk-neutral measure $\mathbb{P}$ and $(A)^+ = \max\{A, 0\}$. Here, $\tau_D^*$ denotes the optimal stopping time, such that the conditional expectation of the right-hand side of (2.2) is given by the maximum value.

Using the standard technique of reformulating an optimal stopping problem into variational inequality, the problem (2.2) can be rewritten as follows.

$$\min\{\mathcal{L}C_{do},\ (S-K)^+\} = 0,$$
$$C_{do}(T,S) = (S-K)^+,\ C_{do}(t,D) = R(t).$$

on domain $\{(t, S)|0 < t \leq T, 0 < S < \infty\}$ and where

$$\mathcal{L} \equiv \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2} + (r-q)S\frac{\partial}{\partial S} - r$$

Let us define the region $\mathcal{D}$ as

$$\mathcal{D} = \{(t,S) \mid 0 \leq t \leq T,\ D < S < \infty\}.$$

Then, the domain $\mathcal{D}$ can be divided into the stopping region $\mathcal{S}_D$ and the continuous region $\mathcal{C}_D$. In terms of the value function $C_{do}(t, S)$, $\mathcal{S}_D$ and $\mathcal{C}_D$ are given by

$$\mathcal{S}_D = \{(t,S) \mid C_{do}(t,S) = (S-K)^+\},$$
$$\mathcal{C}_D = \{(t,S) \mid C_{do}(t,S) > (S-K)^+\}.$$

Then, the boundary that separates $\mathcal{C}_D$ from $\mathcal{S}_D$ is referred to as the *free boundary*, and is given by

$$S_{do}(t) = \inf\{s \in \mathbb{R}^+ \mid (t,s) \in \mathcal{S}_D\},\ t \in [0,T]$$

Similar to American type call options, the stopping region and the continuation region of $C_{do}$ correspond to $S > S_{do}(t)$ and $S \leq S_{do}(t)$, respectively. In terms of the free boundary $S_{do}(t)$, the continuation region $\mathcal{C}_D$ can be expressed by

$$\mathcal{C}_D = \{(t,s) \mid 0 < S < S_{do}(t)\}.$$

Especially, in continuation region $\mathcal{C}_D$, $C_{do}$ satisfies the following partial differential equation (PDE) :

$$\mathcal{L}C_{do}(t,S) = 0,\ \ 0 < S < S_{do}(t).$$

Overall, for the time-reversed value $\bar{C}_{do}(\tau,S) = C_{do}(T-\tau,S)$, $\bar{R}(\tau) = R(T-\tau)$, and $\bar{S}_{do}(\tau) = S_{do}(T-\tau)$ with $\tau = T-t$, we can obtain the following PDE problem :

$$\frac{\partial\bar{C}_{do}}{\partial\tau} = \frac{\sigma^2}{2}S^2\frac{\partial^2\bar{C}_{do}}{\partial S^2} + (r-q)S\frac{\partial\bar{C}_{do}}{\partial S} - r\bar{C}_{do} \tag{2.3}$$

with boundary condition

$$\begin{aligned} &\bar{C}_{do}(0,S) = (S-K)^+\\ &\bar{C}_{do}(\tau,\bar{S}_{do}(\tau)) = \bar{S}_{do}(\tau) - K\\ &\frac{\partial\bar{C}_{do}}{\partial S}(\tau,\bar{S}_{do}) = 1,\ \bar{C}_{do}(\tau,D) = \bar{R}(\tau)\end{aligned} \tag{2.4}$$

### 2.2. American up and out put option

Using approach similar to that of Section 2.1, we derive the PDE with boundary conditions on the American up and out put option price. Let $P_{uo}(t, S)$ denote the price of American up and out put option with upper barrier $U$ such that $S_0 < U$ and $K < U$. For upper barrier $U$, let us define the stopping time $\tau_U$ of $\mathcal{F}_t$,

$$\tau_U = \inf\{t > 0 \mid S_t = U\}.$$

Then, $P_{uo}(t, S)$ satisfies the following optimal stopping problem

$$P_{uo}(t,S) = \max_{t\leq\xi_t\leq T} \mathbb{E}\Big[e^{-r(\xi_t - t)}(K-S_{\xi_t})^+\mathbf{1}_{\{\tau_U\geq T\}} + e^{-r(\xi_t - t)}R(\xi_t)\mathbf{1}_{\{\tau_U<T\}} \mid S_t = S\Big]. \tag{2.5}$$

Similar to the work done in Section 2.1, if we let $\mathcal{U} = \{(t,s) \mid 0 \leq t \leq T,\ 0 < S < U\}$, the domain $\mathcal{U}$ can be divided into the stopping region $\mathcal{S}_U$ and the continuation region $\mathcal{C}_U$, so that $\mathcal{S}_U$ and $\mathcal{C}_U$ can be represented by

$$\mathcal{S}_U = \{(t,S) \in \mathcal{U} \mid P_{uo}(t,S) = (K-S)^+\},$$
$$\mathcal{C}_U = \{(t,S) \in \mathcal{U} \mid P_{uo}(t,S) > (K-S)^+\}.$$

Then, the optimal stopping time $\tau_P^*$ given by (2.5) satisfies $\tau_P^* = \{u \in [t,T] \mid (u, S_u) \in \mathcal{S}_U\}$. By similarly in Section 2.1, we can define the free boundary $S_{up}(t)$ of an American up and out put option as

$$S_{do}(t) = \inf\{t > 0 \mid (t,s) \in \mathcal{S}_U\}.$$