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An optimal consumption, leisure, and investment problem with an option to retire and negative wealth constraints

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ABSTRACT

This paper attempts to choose the optimal consumption, leisure, investment, and voluntary retirement time under the negative wealth constraint. The Dynamic Programming method is used to derive the value function and to identify the optimal policies when the agent's utility function of consumption and leisure is given in the form of Cobb–Douglas. Finally, the effects of negative wealth constraints were discussed by examining the optimal policies that vary depending on the degree of the negative wealth constraint.

1. Introduction

Since the pioneer work of Merton [10,11], the research related to consumption and portfolio selection has been further developed through the dynamic programming method based on Karatzas et al. [6]. In this field, there are special issues regarding a voluntary retirement problem ([1,2,4,5,8,9,12,13] etc), a non-negative wealth constraint ([2,4,5,9] etc), and a negative wealth constraint ([12]). In this paper, we basically develop the results considered by Koo et al. [8] with a negative wealth constraint. In this model, the agent wants to find the optimal consumption, leisure, portfolio, and voluntary retirement time under a negative wealth constraint. We allow a continuum of choice between labor and leisure, and we obtain the value function in three regions similar to Koo et al. [8]. Because of a negative wealth constraint, we can observe other economical implications. We can see that the retirement threshold wealth level decreases as the negative wealth constraint increases. Also, we analyze the sensitivity of results such as optimal investment with respect to wealth constraint.

The rest of the paper is organized as follows. Section 2 describes the basic settings on the economy, Section 3 presents the optimization problem, and derives the optimal policies. Section 4 briefly mentions the numerical results of the related implicit solution and consider the effects of the negative wealth constraint on the retirement threshold wealth level and investment. Finally, Section 5 includes the implications of this work and future research plan.

2. The economy

In the financial market, we assume that there are two tradable assets: One is a riskless asset, which follows $dS_t^0/S_t^0 = rdt$ and the other is a risky asset, which follows $dS_t^1/S_t^1 = \mu dt + \sigma dB_t$, where the parameters $r > 0$, $\mu > r$ and $\sigma > 0$ are assumed to be constants, B_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the augmentation $\mathcal{F}_t$ under $\mathbb{P}$ is generated by the Brownian motion B_t.

Let π_t be the amount of money invested in the risky asset S_t^1, $c_t \geq 0$ be the consumption rate at time t, $l_t \geq 0$ be the leisure rate at time t, and τ be $\mathcal{F}_t$-stopping time, which stands for the time of voluntary retirement. All π_t, c_t and l_t are $\mathcal{F}_t$-measurable processes, which are satisfying, respectively, for all $t \geq 0$ a.s.

$$\int_0^t \pi_s^2 ds < \infty, \quad \int_0^t c_s ds < \infty, \quad \int_0^t l_s ds < \infty.$$

Prior to retirement, l_t is a control variable under the restriction $0 \leq l_t \leq L < \bar{L}$, where L and $\bar{L}$ are assumed to be constants. But during post-retirement, the agent chooses the full leisure $l_t = \bar{L}$, that is, l_t is a constant and is no longer the control variable. We assume that the agent receives a labor income of $w(\bar{L} - l_t) \geq 0$, where $w > 0$ is constant wage rate.

Thus the wealth process X_t of the agent at time t is governed by

$$dX_t = \pi_t \frac{dS_t}{S_t} + (X_t - \pi_t)rdt - c_t dt + w\left(\bar{L} - l_t\right)dt \tag{2.1}$$

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$$= \begin{cases} [rX_t + \pi_t(\mu - r) - c_t + w(\bar{L} - l_t)]dt + \sigma \pi_t dB_t & \text{if } 0 \le t < \tau \\ [rX_t + \pi_t(\mu - r) - c_t]dt + \sigma \pi_t dB_t & \text{if } t \ge \tau \end{cases}, \quad X_0 = x, \tag{2.2}$$

where stock investment π_t and bond investment $X_t - \pi_t$ in (2.1) can be negative (see Remark 2.3 of Karatzas et al. [7]).

In this work, we use a Cobb-Douglas utility function for consumption and leisure as follows:

$$u(c_t, l_t) := \frac{1}{\alpha} \frac{\left(c_t^{\alpha} l_t^{1-\alpha}\right)^{1-\gamma}}{1-\gamma}, \quad 0 < \alpha < 1 \text{ and } \gamma > 0 \ (\gamma \neq 1), \tag{2.3}$$

where γ is the agent's coefficient of relative risk aversion and α is a constant weight for consumption. If we define $\gamma_1 := 1 - \alpha(1-\gamma)$, then the utility function (2.3) can be rewritten as

$$u(c_t, l_t) = \frac{c_t^{1-\gamma_1} l_t^{\gamma_1 - \gamma}}{1-\gamma_1} \quad \text{if } 0 \le t < \tau$$

and

$$u(c_t, \bar{L}) = \frac{c_t^{1-\gamma_1} \bar{L}^{\gamma_1 - \gamma}}{1-\gamma_1} \quad \text{if } t \ge \tau.$$

$$V(x) = \begin{cases} \left(\frac{w\alpha}{1-\alpha}\right)^{-(\gamma_1-\gamma)} \left[\frac{r - \frac{1}{2}\theta^2 m_+}{\rho} A_1 \zeta^{-\gamma(m_+ + 1)} + \frac{r - \frac{1}{2}\theta^2 m_-}{\rho} A_2 \zeta^{-\gamma(m_- + 1)} + \frac{1}{K(1-\gamma_1)} \zeta^{1-\gamma} \right] & \text{if } -\nu \frac{w\bar{L}}{r} \le x < \tilde{x} \\ L^{\gamma_1 - \gamma} \left[\frac{r - \frac{1}{2}\theta^2 m_+}{\rho} B_1 \eta^{-\gamma_1(m_+ + 1)} + \frac{r - \frac{1}{2}\theta^2 m_-}{\rho} B_2 \eta^{-\gamma_1(m_- + 1)} + \frac{1}{K_1(1-\gamma_1)} \eta^{1-\gamma_1} \right] & \text{if } \tilde{x} \le x < \bar{x} \\ \frac{\bar{L}^{\gamma_1 - \gamma}}{K_1^{\gamma_1}(1-\gamma_1)} x^{1-\gamma_1} & \text{if } x \ge \bar{x} \end{cases} \tag{3.4}$$

3. The optimization problem

In this work, the agent wants to maximize her lifetime expected utility. Thus the maximization problem can be represented as follows:

$$\begin{aligned} V(x) &= \sup_{(c,l,\pi,\tau) \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t, l_t) dt \right] \\ &= \sup_{(c,l,\pi,\tau) \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\rho t} \frac{c_t^{1-\gamma_1} l_t^{\gamma_1-\gamma}}{1-\gamma_1} dt \right. \\ &\quad \left. + \bar{L}^{\gamma_1-\gamma} \int_\tau^\infty e^{-\rho t} \frac{c_t^{1-\gamma_1}}{1-\gamma_1} dt \right] \\ &= \sup_{(c,l,\pi,\tau) \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\rho t} \frac{c_t^{1-\gamma_1} l_t^{\gamma_1-\gamma}}{1-\gamma_1} dt + e^{-\rho \tau} U(X_\tau) \right], \end{aligned} \tag{3.1}$$

where $\rho > 0$ is a subjective discount factor, $\mathcal{A}(x)$ is an admissible class of (c, l, π, τ) and $U(x)$ is a post-retirement value function (see Theorem 3.1), subject to the budget constraint (2.2) and the negative wealth constraint

$$X_t \ge -\nu \frac{w\bar{L}}{r} > -\frac{w\bar{L}}{r}, \quad \text{for } t \ge 0 \text{ and } \nu \in [0, 1). \tag{3.2}$$

The negative wealth constraint (3.2) allows that the agent can borrow against partial portion of the future labor income, that is, if $\nu \uparrow 1$, then she can borrow fully against future labor income, but, if $\nu = 0$, then she cannot borrow against future labor income, which is called the non-negative wealth constraint or the borrowing constraint.

Remark 3.1. As mentioned below of (2.2), π_t and $X_t - \pi_t$ can be negative. Because of the negative wealth constraint (3.2), the wealth X_t (sum of the stock investment π_t and the bond investment $X_t - \pi_t$), however, is always bounded below by $-\nu \cdot w\bar{L}/r$.

The following assumption always holds throughout this paper.

Assumption 3.1.

$$K := r + \frac{\rho - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2 > 0$$

and

$$K_1 := r + \frac{\rho - r}{\gamma_1} + \frac{\gamma_1 - 1}{2\gamma_1^2}\theta^2 > 0,$$

where $\theta := (\mu - r)/\sigma$ is a market price of risk.

Remark 3.2. For later consideration, we define the quadratic equation

$$f(m) := \frac{1}{2}\theta^2 m^2 + \left(\rho - r + \frac{1}{2}\theta^2\right) m - r = 0 \tag{3.3}$$

with two roots $m_+ > 0$ and $m_- < -1$.

Now we use the dynamic programming approach to derive the value function of the optimization problem (3.1).

Theorem 3.1. *The value function is given by*

where $\tilde{x}$ is the wealth level corresponding to the consumption level $\tilde{c}$ at the leisure $l = L$, $\bar{x}$ is the threshold wealth level corresponding to the consumption level $\bar{c}$ at retirement time τ with $0 < \tilde{x} < \bar{x}$, and $\hat{c}$ is the consumption level corresponding to the negative wealth constraint.

$$\tilde{c} = \frac{w\alpha L}{1-\alpha} > 0$$

and

$$\tilde{x} = A_1 \left(\frac{w\alpha L}{1-\alpha}\right)^{-\gamma m_+} + A_2 \left(\frac{w\alpha L}{1-\alpha}\right)^{-\gamma m_-} + \frac{wL}{(1-\alpha)K} - \frac{w\bar{L}}{r}.$$

$\zeta > 0$ and $\eta > 0$ are the solutions to the following algebraic equations, respectively,

$$x = A_1 \zeta^{-\gamma m_+} + A_2 \zeta^{-\gamma m_-} + \frac{1}{\alpha K}\zeta - \frac{w\bar{L}}{r}$$

and

$$x = B_1 \eta^{-\gamma_1 m_+} + B_2 \eta^{-\gamma_1 m_-} + \frac{1}{K_1}\eta - \frac{w(\bar{L} - L)}{r}. \tag{3.5}$$

The coefficients A_1, A_2, B_1 and B_2 are given as follows:

$$A_1 = -\frac{\gamma m_- \frac{w\bar{L}}{r}(1-\nu) - \frac{1+\gamma m_-}{\alpha K}\hat{c}}{\gamma(m_+ - m_-)} \hat{c}^{\gamma m_+},$$

$$A_2 = \frac{\gamma m_+ \frac{w\bar{L}}{r}(1-\nu) - \frac{1+\gamma m_+}{\alpha K}\hat{c}}{\gamma(m_+ - m_-)} \hat{c}^{\gamma m_-}, \tag{3.6}$$