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Is a hyperchaotic attractor superposition of two multifractals?

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ABSTRACT

In the context of chaotic dynamical systems with exponential divergence of nearby trajectories in phase space, hyperchaos is defined as a state where there is divergence or stretching in at least two directions during the evolution of the system. Hence the detection and characterization of a hyperchaotic attractor is usually done using the spectrum of Lyapunov Exponents (LEs) that measure this rate of divergence along each direction. Though hyperchaos arise in different dynamical situations and find several practical applications, a proper understanding of the geometric structure of a hyperchaotic attractor still remains an unsolved problem. In this paper, we present strong numerical evidence to suggest that the geometric structure of a hyperchaotic attractor can be characterized using a multifractal spectrum with two superimposed components. In other words, apart from developing an extra positive LE, there is also a structural change as a chaotic attractor makes a transition to the hyperchaotic phase and the attractor changes from a simple multifractal to a dual multifractal, equivalent to two inter-mingled multifractals. We argue that a cross-over behavior in the scaling region for computing the correlation dimension is a manifestation of such a structure. In order to support this claim, we present an illustrative example of a synthetically generated set of points in the unit interval (a Cantor set with a variable iteration scheme) displaying dual multifractal spectrum. Our results are also used to develop a general scheme to generate both hyperchaotic as well as high dimensional chaotic attractors by coupling two low dimensional chaotic attractors and tuning a time scale parameter.

1. Introduction

A dynamical system is considered to be chaotic if it shows the property of sensitive dependence on initial conditions. For such systems, two nearby trajectories diverge exponentially in time during the evolution of the system, indicating that one of the LEs is positive. Hyperchaos is formally defined as a state where there is divergence in at least two directions as the system evolves. Hyperchaotic attractors are thus characterized by at least two positive LEs and are considered to be much more complex in terms of topological structure and dynamics compared to low dimensional chaotic attractors. In the last two decades, hyperchaotic systems have attracted increasing attention from various scientific and engineering communities due to a large number of practical applications. These include secure communication and cryptography [1,2], synchronization studies using electro-optic devices [3,4] and as a model for chemical reaction chains [5]. In all these applications, the complexity of the underlying attractor has a major role to play.

Though the concept of hyperchaos was introduced many years ago [6], a systematic understanding of the topological and fractal structure of the attractors generated from the hyperchaotic systems is lacking till date. Studies in this direction have been very few except a series of papers on a system of unidirectionally coupled oscillators [7–9] in which the authors have discussed many aspects of the structure and transition to hyperchaos in the model, including dual scaling regions in the hyperchaotic phase.

Hyperchaotic attractors generated by continuous systems are, in general, higher dimensional with the fractal dimension $D_0 > 3$ and trajectories diverging in at least two directions as the system evolves in time. Hence the detection of hyperchaos is generally done using the LEs with the transition to hyperchaos marked by the crossing of the second largest LE above zero. One of our aims in this paper is to try and get a more quantitative information regarding the structure of the hyperchaotic attractor in terms of the spectra of dimensions and use this information to detect the transition to hyperchaos.

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Recently, we have done a detailed dimensional analysis [10,11] of several standard hyperchaotic models and have established some results which are common to all these systems. We have applied a modified box counting scheme and have obtained an improved scaling region for computing the fractal dimension of the system. Also, we have shown that the topological structure of the underlying attractor changes suddenly as the system makes a transition from chaos to hyperchaos and there is a cross-over behavior in the scaling of the correlation dimension D_2 resulting in two different scaling regions in the hyperchaotic phase. Here we investigate this cross-over behavior in more detail numerically and show that we can derive the whole spectrum of D_q values corresponding to the two different scaling regions. We consider this result as a consequence of the fact that the geometric structure of a hyperchaotic attractor is equivalent to that of two inter-mingled multifractals and the cross-over property is a manifestation of this structure. In other words, the overall fractal structure of a hyperchaotic attractor can be characterized by two superposed $f(\alpha)$ spectrums.

It should be noted that the *multiscales* exhibited by multifractals have recently become an interesting area of research and have been discussed in various contexts. For example, the importance of multiscale multifractal analysis (MMA) has been demonstrated in the study of human heart rate variability time series [12], where the multifractal properties of the measured signal depends on the time scale of fluctuations or the frequency band. Also, multiscale multifractal intermittent turbulence in space plasmas has been investigated in the time series of velocities of solar wind plasma [13]. In order to convince the reader that a dual multifractal structure can be realized in practice, we generate a Cantor set using variable iteration scheme which displays dual slopes in the scaling region. Finally, the specific information regarding the structure of the hyperchaotic attractor provides us the possibility of generating hyperchaos by coupling two chaotic attractors, a result already shown in the literature [14,15]. Here we present a general scheme for this to get both hyperchaos and high dimensional chaos by varying a control parameter.

Our paper is organized as follows: In the next section, we present a brief summary of the standard multifractal approach for a point set. In Section 3, we discuss the details of numerical computations of the multifractal spectrum to show how the structure of a hyperchaotic attractor varies from that of an ordinary chaotic attractor. In order to validate our arguments regarding the structure of the hyperchaotic attractor, we present an example of a system having analogous structure in Section 4 which is a synthetically generated Cantor set using a specific iterative scheme. The details regarding the generation of hyperchaos based on our numerical results are discussed in Section 5. The paper is concluded in Section 6.

2. Mathematical preliminaries

It is well known that, unlike ideal fractals, real world systems and limited point sets exhibit self similarity only over a finite range of scales [16]. Thus in the present case, statistical self similarity and hence the multifractal behavior changes between two finite range of scales. Multifractality is commonly related to a probability measure that can have different fractal dimensions on different parts of the support of this measure. Many authors have discussed the standard multifractal approach in detail [17–20] and we briefly summarise the main results below for a point set (such as, an attractor generated by a chaotic system).

Let the attractor be partitioned into M dimensional cubes of side r, with $N(r)$ being the number of cubes required to cover the attractor. If $p_i(r)$ is the probability that the trajectory passes through the ith cube, then $p_i(r) = N_i/N_p$, where N_i is the number of points in the ith cube and N_p the total number of points on the attractor. We now assume that $p_i(r)$ satisfies a scaling relation

$$p_i(r) \propto r^{\alpha_i} \tag{1}$$

where α_i is the scaling index for the ith cube. We now ask how many cubes have the same scaling index α_i or have scaling index within α and $\alpha + d\alpha$ (if α is assumed to vary continuously). Let this number, say $g(\alpha)d\alpha$, scales with r as

$$g(\alpha) \propto r^{-f(\alpha)} \tag{2}$$

where $f(\alpha)$ is a characteristic exponent. Obviously, $f(\alpha)$ behaves as a dimension and can be interpreted as the fractal dimension for the set of points with scaling index α. This also implies that the attractor can be characterized by a spectrum of dimensions normally denoted by D_q (where q can, in principle, vary from $-\infty$ to ∞) [21], that can be related to $f(\alpha)$ through a Legendre transformation [22]. The plot of $f(\alpha)$ as a function of α gives a one hump curve with maximum corresponding to D_0, the simple box counting dimension of the attractor.

Note that, in the above arguement, the scaling exponent α measures how fast the number of points within a box decreases as r is reduced. It therefore measures *the strength of a singularity* for $r \to 0$. For a realistic attractor, with limited number of data points, the limit $r \to 0$ is not accessible and hence one chooses a suitable scaling region for r to compute α and $f(\alpha)$. This is where a hyperchaotic attractor becomes different from an ordinary chaotic attractor, as per our numerical results. We find that, to characterize the multifractal structure of a hyperchaotic attractor, two seperate scaling regions are to be considered indicative of the presence of two underlying multifractals. The detailed numerical results are presented in the next section.

3. Hyperchaotic attractor as a dual multifractal

Before going into the computation of the multifractal spectrum, we discuss very briefly our results on D_2 obtained using the modified box counting scheme [10,11], where the scaling region for computing D_2 is fixed algorithmically. The attractor is covered using M-dimensional cubes of size r. The probability p_i that the trajectory passes through the ith cube is computed by taking an ensemble average of the number of points falling in the ith cube. This modifies the equation for computing the weighted box counting sum $B(r)$ as:

$$B(r) = \frac{1}{N_p^2}\left[\sum_i m_i^2 - N_p\right] \tag{3}$$

where N_p is the total number of points on the attractor and m_i is the number of points falling in the ith box. The correlation dimension is then calculated as the scaling index of the variation of $B(r)$ with r as:

$$D_2 \equiv \lim_{r\to 0} \log B(r)/\log(r) \tag{4}$$

For the present study, we use time series from three standard hyperchaotic systems, namely, the Chen hyperchaotic flow [23], the Mackey–Glass (M-G) time delayed system [24] and the Ikeda time delayed system [25]. For the hyperchaotic flow, we fix the parameters as studied in detail in [10] ($a = 35, b = 4.9, c = 25, d = 5, e = 35, k = 100$) to generate the hyperchaotic time series. For M-G and Ikeda systems, we use the time delay τ as the control parameter with the other parameters fixed as $\beta = 2, \gamma = 1, n = 10$ for M-G and $a = 5, m = 20$ for Ikeda respectively. We have studied the transition to hyperchaos in these two time delayed systems in detail [11] and here we choose $\tau = 6.40$ for M-G and $\tau = 0.56$ for Ikeda for generating the hyperchaotic time series.