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Chaos suppression in fractional systems using adaptive fractional state feedback control

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ABSTRACT

This paper studies an adaptive Fractional Order (FO) state feedback control and the synchronization of FO chaotic systems. In the proposed algorithm, the feedback gain follows a FO adaptive control law. Two different control procedures are introduced namely full- and reduced-state feedback controllers, with single and vector variable feedback gains, respectively. The stability analysis is provided by means of the FO Lyapunov theorem both for the control and synchronization problems. Three examples are given to illustrate the effectiveness of the proposed scheme.

1. Introduction

Fractional Calculus (FC) is a branch of mathematical analysis and its applications that attracted considerable attention during the two last decades [1]. FC generalizes the concept of derivative or integral of a function to a non-integer order [2]. The models that comprise fractional derivatives and integrals, the so called fractional-order (FO) systems, are recognized to be an effective tool for modeling many physical processes [3]. In fact, distinct phenomena are successfully modeled in the FO perspective, such as earthquakes [4], the Rayleighs piston [5], stock markets [6], pendula [7], muscular blood vessels [8], and electromagnetism [9].

Recent results reported in the literature show that FO systems can behave chaotically [10], such as in the FO hyperchaotic Lorenz [11] and Chen [12], reverse butterfly-shaped chaotic [13], chaotic Arneodo [14] and unified [15], and FO complex power [16] systems.

The nonlinear chaotic systems have special characteristics such as unpredictable evolutions and strong dependence on the initial conditions [17]. FO chaos suppression including control and synchronization became a relevant topic during the past decade due to their potential application in different areas [13,18–22]. Several control algorithms have been extensively investigated in the literature to deal with the chaos suppression, such as the active [23–25], predictive [26], sliding-mode [27–29], robust [30,31] and adaptive [16,32–37] control methods.

The key challenges in the control systems are the selection of the architecture and the parameter tuning. In the area of chaotic systems, a simple accessible controller is particularly significant both in the theoretical and practical perspectives. Among the proposed algorithms, the state feedback controller is as a promising technique to achieve these goals. The main design topic in the state feedback control is the selection of the feedback gains that affect directly the stability of the closed-loop system.

There are two strategies, such as the adaptive and non-adaptive algorithms, to implement the gain of state feedback controller by using integer or FO Lyapunov functions. Furthermore, the adaption law of the feedback gain can be integer or fractional. In [16], a switching controller with FO adaption law was introduced to stabilize FO systems based on the FO stability theory (i.e. using FO Lyapunov functions). In [34], an adaptive controller with integer adaption law for synchronization of a class of three-dimensional FO chaotic systems was presented using an integer Lyapunov function. In [35], a single state feedback controller with integer adaption law was proposed to stabilize a three-dimensional FO chaotic system using integer Lyapunov stability theory.

With these ideas in mind, the suppression of chaos in FO systems is investigated here considering, (1) a systematic technique for designing adaptive state feedback control of FO chaotic systems with FO law and FO Lyapunov stability functions, (2) two scenarios with full and reduced-state feedback controllers both in a single and vector variable feedback gains, (3) a strategy for handling the problem both in the stability analysis and design phases. In this perspective, the paper is organized as follows. Section 2 recalls some fundamental concepts and results of the FC

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theory. Sections 3 and 4 develop the main concepts for FO adaptive control and synchronization of FO chaotic systems, respectively. Section 5 analyzes several simulations that demonstrate the effectiveness of tproposed control technique. Finally, Section 6 discusses the results and outlines the conclusions.

2. Fundamental concepts and results

In this section, the fundamental definitions of fractional operators are recalled. Some properties and lemma related to the stability analysis of FO systems are given below.

Definition 1 [36]**.** The Caputo fractional derivative of order α of a function $x(t) \in C^{n+1}$ $([t_0, +\infty), R)$ is defined as:

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{1}$$

where $\boldsymbol{t > t_0}$, $n-1 < \alpha < n \in Z^+$ and $\Gamma(\cdot)$ is the gamma function, $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

In particular, if $0 < \alpha < 1$, then

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} x^{(1)}(\tau) d\tau, \tag{2}$$

Definition 2 [36]**.** The Riemann–Liouville fractional integral of order α of a continuous function $x(t) \in C^{n+1}([t_0, +\infty), R)$ is defined as

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} x(\tau) d\tau, \tag{3}$$

where $t \geq t_0$ and $\alpha > 0$.

Definition 3 [36]**.** Let $f(t)$ be a continuously differentiable function defined on $x(t) \in C^n[a, b]$, then

$$I^\alpha D^\alpha x(t) = x(t) - \sum_{k=1}^{n-1} \frac{x^{(k)}(a)}{k!} (t-a)^k. \tag{4}$$

In particular, if $0 < \alpha < 1$ and $x(t) \in C^1[a, b]$, then

$$I^\alpha D^\alpha x(t) = x(t) - x(a). \tag{5}$$

Definition 4 [38]**.** Let $X(t) \in R^n$ be a continuously differentiable function, then

$$D^\alpha \left(\frac{1}{2} X^T(t) X(t) \right) \leq X^T(t) D^\alpha X(t), \tag{6}$$

where $t \geq t_0$ and $\alpha \in (0, 1)$.

Corollary 1. *From Definitions 1 and 4, for any continuously differentiable function $x(t)$ and any constant $\rho \in R$, it can be concluded that we have*

$$\begin{aligned} D^\alpha (x(t)-\rho)^2 &= D^\alpha \left(x^2(t) - 2\rho x(t) + \rho^2 \right), \\ &\leq 2(x(t)-\rho) D^\alpha x(t), \end{aligned} \tag{7}$$

where $0 < \alpha < 1$ and $t \in [0, \infty)$.

3. Main concepts for FO adaptive control of FO chaotic system

The success of linear state feedback relies on an appropriate choice of the feedback gains. In this section, we develop two distinct procedures for determining the feedback gains for FO chaotic systems.

Consider the FO chaotic system described by

$$D^\alpha X(t) = f(X), \tag{8}$$

where $0 < \alpha < 1$, $X = (x_1, x_2, \cdots, x_n)^T \in R^n$ and $f(X) = (f_1(X), f_2(X), \cdots, f_n(X))^T : R^n \to R^n$ is a nonlinear vector function.

Assume that $\Omega \subset R^n$ is a chaotic bounded set of Eq. (8), which is globally attractive, and that $X = X^*$ is an equilibrium point of the system (8) in Ω. Having in mind the controller design and its stability analysis, we assume that $\forall X = (x_1, x_2, \cdots, x_n)^T \in \Omega$ and that there is a constant $l = [l_1, \cdots, l_n] > 0$ such that

$$|f_i(X)| \leq l_i |X|_\infty, \qquad i = 1, 2, \cdots, n, \tag{9}$$

where $|X|_\infty$ denotes the ∞-norm of X, defined as $|X|_\infty = \max_j |X_j|$, $j = 1, 2, \cdots, n$. This condition is easily met and is valid for the all well-known finite dimensional chaotic and hyper chaotic systems.

With the above assumptions two control scenarios, namely full- and reduced-state feedback control, are proposed to deal with the problem under study and designed separately. Without loss of the generality, we suppose that origin is an equilibrium point of the system (8).

3.1. Full state feedback control

In this scenario, in order to stabilize the system (8) toward its equilibrium point $X^* = 0$, we first consider the FO adaptive state feedback controller with a single variable feedback gain

$$U(t) = -k_1(t) X(t) = -k_1(t)[x_1(t), x_2(t), \cdots, x_n(t)]^T, \tag{10}$$

under the update law

$$D^\alpha k_1(t) = \gamma \sum_{j=1}^{n} \left(x_j(t) - x_j^* \right)^2 = \gamma X^T(t) X(t) \tag{11}$$

is introduced, where $k_1(t) \geq 0$ is the feedback gain, γ is a positive constant and x_j^* is *jth* row of X^* vector. Using the adaptive state controller (10), the overall closed-loop system is obtained as

$$\begin{aligned} D^\alpha X(t) &= f(X) + U(t) = f(X(t)) - k_1(t)(X(t) - X^*), \\ &= f(X(t)) - k_1(t) X(t). \end{aligned} \tag{12}$$

Theorem 1. *Consider the closed-loop system (12) with arbitrary initial values. Using the adaptive feedback controller given in Eq. (10) and (11), the closed-loop system is stable and the chaotic orbits $(X(t), k_1(t))^T$ converge to $\left(0, k_1^*\right)^T$ when $t \to \infty$, where k_1^* is a positive constant depending on the the its initial values and γ.*

Proof. Construct the following candidate Lyapunov function

$$V(t) = \frac{1}{2} X^T(t) X(t) + \frac{1}{2} \frac{1}{\gamma} (k_1(t) - L)^2, \tag{13}$$

where L is a sufficiently large positive constant, such that $L \geq \sum_{i=1}^{n} l_i$. By differentiating the Lyapunov function along the trajectories, it yields

$$\begin{aligned} D^\alpha V(t) &\leq X^T(t) D^\alpha X(t) + \frac{1}{\gamma} (k_1(t) - L) D^\alpha k_1(t), \\ &\leq X^T(t)(f(X(t)) - k_1(t) X(t)) \\ &\quad + (k_1(t) - L) X^T(t) X(t), \\ &\leq X^T(t) f(X(t)) - L X^T(t) X(t), \\ &\leq \left(\sum_{i=1}^{n} l_i \right) X^T(t) X(t) - L X^T(t) X(t) \leq 0, \end{aligned} \tag{14}$$

which implies that the closed-loop system is stable in the sense of Lyapunov.

In the follow-up, we develop the controller design by choosing different feedback gains, which is so-called full-state feedback controller with vector variable feedback gain. Let us consider the system (8) with $\gamma = [\gamma_1, \gamma_2, \cdots, \gamma_n]$ and $K(t) = [k_1(t), k_2(t), \cdots, k_n(t)]$ under the feedback controller