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Nondeterministic basin of attraction

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ABSTRACT

Switching dynamical systems occur frequently in many areas of physics and engineering. In this paper we consider a piecewise linear map, that randomly switches in between more than one different functional forms, in any one of the compartments of the phase space. We establish that for such kind of maps there exists a region in the phase space consisting of a special property that, the dynamics of any orbit starting from any particular point, lying inside this region is not deterministic, as any two orbits may find different destinations despite of starting from the same initial point. In other words, even if two orbits start from the same initial point (belonging to the specified region in the phase space), then also they may not converge or diverge together, i.e., one of them may converge to a stable fixed point whereas the other one may diverge to infinity.

1. Introduction

Switching dynamical systems frequently occur in the context of many physical, biological, engineering and control systems, systems involving robotics etc. In general we observe mainly two kinds of switching among these systems—state dependent and time dependent switching. Piecewise smooth(PWS) maps are very common examples of state dependent switching dynamical systems. An n dimensional piecewise smooth map is described by two different functional forms in two different compartments of the phase space, separated by an $(n-1)$ dimensional subspace called 'border'. The map is everywhere differentiable except at the border. These type of maps are used to model many systems in electrical and mechanical engineering, biology etc. For example, switching in power electronics and grazing in mechanical systems are a couple of common phenomenon which give rise to these kind of PWS maps. In general, the discrete time representation of any continuous time hybrid dynamical system, is described by a PWS maps. These maps may either be continuous (PWS continuous maps) or discontinuous (PWS discontinuous maps) at the border. In the literature we find several systems which are modelled by PWS continuous as well as PWS discontinuous maps [1–8]. A lot of intense researches have been carried out to study the various dynamical properties of these types of systems [9–17].

On the other hand there are systems, involving only time dependent switching also. Iterated function systems are very common examples of such kind of systems. It has been proved that a large variety of fractals are nothing but attractors of a class of iterated function systems [18,19]. Several other kinds of time dependent switching systems, arising due the evolution of the system dynamics in a changing environment have also been studied by Penrose et al. , where many dynamical properties, such as invariant density etc. have been investigated [20]. Besides the two types of switching phenomena discussed above, there may arise many situations, where we may need both time and state dependent switching to model a system. These types of systems are used, in order to tackle some specific practical situations. Such kind of systems has been studied already. It has been shown that an invariant attractor may bifurcate into a non-invariant attractor due to border-collision bifurcation in these type of systems [21].

Basin of attraction of an attractor is defined as a region of the phase space, with the property that, any orbit which starts inside this region, ultimately converges to the attractor. Moreover if an orbit starts from any point lying outside this basin of attraction, then it certainly does not converge to that attractor, it either converges to some other attractor or it simply diverges. Therefore any two orbits, which start from a particular point in the phase space, either converge or diverge together. In this paper we have shown that the above situation may not hold true for a type of piecewise smooth maps that randomly switches in between one or more different functional forms. Here we consider a one dimensional piecewise linear continuous map, the functional form of which switches in between two different functions depending upon time, in any one of the compartments of the phase space. Therefore the system contains state dependent as well as time dependent switching also. The time dependent switching is assumed here to be random, following some specific probability distribution. We have showed that, in case of these types of systems, there exist a region in the

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phase space with the property that, if any two orbits start from a same point inside this region, then one of them may converge to a stable fixed point whereas the other may diverge to infinity. Therefore each orbit has non-zero probabilities of convergence (to a stable fixed point) as well as divergence (to infinity), even if they start from the same initial point.

2. Mathematical formulation

Consider the one dimensional piecewise smooth map given by the following

$$x_{n+1} = \begin{cases} \lambda + k_1 x_n & : x \le 0 \\ \lambda + k x_n & : x \ge 0 \end{cases} \tag{1}$$

Where $0 < k_1 < 1$ and $k > 1$. Then when $\lambda < 0$ there exist two fixed points, one to the left of the border, say x_L and the other to the right side of the border, say x_R. Here

$$x_L = \frac{\lambda}{1-k_1}; \quad x_R = \frac{\lambda}{1-k} \tag{2}$$

We notice that x_R is an unstable fixed point whereas x_L is a stable fixed point with $(-\infty, x_R)$ as its basin of attraction.

Now consider that the slope of the right hand side of the piecewise smooth map, i.e., k is not a constant, rather it varies depending upon time randomly, according to a specified probability distribution. In that case the map (1) becomes

$$x_{n+1} = \begin{cases} \lambda + k_1 x_n & : x \le 0 \\ \lambda + k(n) x_n & : x \ge 0 \end{cases} \tag{3}$$

where

$$k(n) = \begin{cases} k_2 & : \text{with probability } p_1 \\ k_3 & : \text{with probability } p_2 \end{cases} \tag{4}$$

and $p_1 + p_2 = 1$, i.e., at any arbitrary n-th time step we have two possibilities (either $k(n) = k_2$ or $k(n) = k_3$) and their probabilities are given as follows.

$$P(k(n) = k_2) = p_1$$

and

$$P(k(n) = k_3) = p_2$$

Now in map (3) if $k(n) = k_2$, then we have the stable fixed point $x_L = \frac{\lambda}{1-k_1}$, with $(-\infty, x_{R_1})$ as its basin of attraction, where $x_{R_1} = \frac{\lambda}{1-k_2}$ is the unstable fixed point of the map. Similarly if $k(n) = k_3$, then we have the stable fixed point $x_L = \frac{\lambda}{1-k_1}$, with $(-\infty, x_{R_2})$ as its basin of attraction, where $x_{R_2} = \frac{\lambda}{1-k_3}$ is the unstable fixed point of the map.

Therefore at any particular instant the stable fixed point of the map (3) remains unaltered, but its basin of attraction changes depending upon the value of $k(n)$. If we denote the basin of attraction of x_L by B then we have the following two probabilities,

$$P(B = (-\infty, x_{R_1})) = p_1$$

and

$$P(B = (-\infty, x_{R_2})) = p_2$$

Therefore in this case we have a piecewise linear map, the functional form of which switches in between more than one functions depending upon time randomly, according to some specified probability distribution. We now determine the basin of attraction for the stable fixed point x_L of the map (3). We have $x_{R_1} = \frac{\lambda}{1-k_2}$ and $x_{R_2} = \frac{\lambda}{1-k_3}$ as two possible unstable fixed point at any instant of time. Then either $x_{R_1} < x_{R_2}$ or $x_{R_1} > x_{R_2}$(we do not consider the possibility that $x_{R_1} = x_{R_2}$ as $k_2 \ne k_3$). Without loss of generality let us assume that $x_{R_1} < x_{R_2}$. Then first consider the region $(-\infty, x_{R_1})$ of the phase space. Since it is the common region of the two possible basin of attractions, any orbit of the map (3) starting inside this region necessarily converges to the stable fixed point x_L. Whereas it is obvious that any orbit starting outside the interval $(-\infty, x_{R_2})$ diverges to infinity. Now the question is—what can we say about the interval (x_{R_1}, x_{R_2}) of the phase space. What will be the future dynamics of any orbit starting inside this region? Will it converge to the stable fixed point or diverge to infinity? Our main aim will now be to investigate these questions.

Before we proceed further, let us discuss a couple of issues. If $|k(n)| < 1$, then there exist contracting linear maps on both the compartments and as long as $\lambda < 0$, both x_{R_1} and x_{R_2} do not exist, whereas x_L still remains as a fixed point of (3). Therefore starting from any initial point, an orbit of (3) converges to the fixed point x_L. Again if $k(n) < -1$, then as long as $\lambda < 0$, x_L exist as a fixed point of (3) on the left hand side compartment, whereas both x_{R_1} and x_{R_2} do not exist. In that case, due to the presence of expanding linear maps with negative slope on the right hand side compartment, all but finite number of iterates of any orbit lie to the left hand side compartment of the phase space and hence the resulting orbit converges to the fixed point x_L. Therefore in both cases x_L remains as the stable fixed point of (3) and the whole phase space serves as its basin of attraction.

3. Determination of basin of attraction

First we notice that there can be three possibilities regarding an orbit of the map (3), which starts inside the interval (x_{R_1}, x_{R_2}).

- After a finite number of iterations the orbit enters the region $(-\infty, x_{R_1})$. In that case the orbit ultimately converges to the stable fixed point x_L.
- After a finite number of iterations the orbit enters the region (x_{R_2}, ∞). In that case the orbit diverges to infinity.
- All the iterates remain confined in the region (x_{R_1}, x_{R_2}).

Let the probabilities of the first and second possibilities are P_{x_0} and $P^*_{x_0}$ respectively. We show that the third possibility is redundant, i.e. the probability of the third possibility is zero and therefore

$$P_{x_0} + P^*_{x_0} = 1$$

In map (3), let us denote the right hand side expanding linear map corresponding to $k = k_2$ and $k = k_3$ by f_{k_2} and f_{k_3} respectively. Suppose we start from any initial point $x_0 \in (x_{R_1}, x_{R_2})$ and for next n number of iterates it remains confined in the same interval (x_{R_1}, x_{R_2}) i.e. $x_n \in (x_{R_1}, x_{R_2})$, for $i = 0, 1, 2, \cdots, n$. Then $x_n = f_n \cdots f_2 f_1(x_0)$, i.e. we apply the sequence of functions $(f_1, f_2, \cdots, f_n)$ to reach x_n from x_0, where each f_i is either f_{k_2} or f_{k_3} as there can be two possibilities at every step that either f_{k_2} or f_{k_3} is applied to the previous iterate, depending upon the relative positions of the iterates and the border. So there exist 2^n number of such possible sequences $(f_1, f_2, \cdots, f_n)$. Denote these 2^n number of sequences as F_i, $i = 1, 2, \cdots, 2^n$, i.e., $F_i \equiv (f_{i_1} = f_1, f_{i_2} = f_2, \cdots, f_{i_n} = f_n)$ where either $f_{i_k} = f_{k_2}$ or $f_{i_k} = f_{k_3}$ for $k = 1, 2, \cdots, n$.

Then we have

$$P(x_n \in (x_{R_1}, x_{R_2})) = \sum_{i=1}^{2^n} P(x_n \in (x_{R_1}, x_{R_2}) | F_i) P(F_i) \tag{5}$$

Where $P(x_n \in (x_{R_1}, x_{R_2}) | F_i)$ is the conditional probability of the fact that $x_n \in (x_{R_1}, x_{R_2})$ subject to the condition that F_i is applied on x_0.

Now we note that

$$\begin{aligned} P(F_i) &= P(f_{i_1} = f_1, f_{i_2} = f_2, \cdots, f_{i_n} = f_n) \\ &= P(f_{i_1} = f_1) P(f_{i_2} = f_2) \cdots P(f_{i_n} = f_n) \end{aligned}$$