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Existence, uniqueness and asymptotic behavior of traveling wave fronts for a generalized Fisher equation with nonlocal delay

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ABSTRACT

This paper is concerned with existence, uniqueness and asymptotic behavior of traveling wave fronts for a generalized Fisher equation with nonlocal delay. The existence of traveling wave fronts is established by linear chain trick and geometric singular perturbation theory. The strategy is to reformulate the problem as the existence of a heteroclinic connection in $\mathbb{R}^4$. The problem is then tackled by using Fenichel's invariant manifold theory. The asymptotic behavior and uniqueness of traveling wave fronts are also obtained by using standard asymptotic theory and sliding method.

1. Introduction

Traveling wave solutions of nonlinear reaction-diffusion equations with delays have been extensively investigated due to the significant applications in several subjects [1,2]. In recent years, a great number of theoretical issues concerning reaction-diffusion equations with spatially and temporally nonlocal terms in the form of the convolution of a kernel with the dependable variable have received considerable attention. This type of equations are believed to be more realistic than the usual kind of reaction-diffusion models for certain population dynamics since populations take time to move in space and generally were not at the position in space at previous times in a population model and thus the time delay term involves a weighted spatio-temporal average over the whole spatial domain and the whole previous times [9]. In particular, the traveling wave solutions to local-delayed (or time-delayed) reaction-diffusion equations [3–7], nonlocal-delayed (or spatio-temporal delayed) reaction-diffusion equations [8–22] have been widely studied. In these references, results on the persistence of the traveling wave fronts for Fisher and generalized Fisher equations have been established for various particular reaction functions and kernels [12,13,15,16] etc.

More precisely, Gourley [12] investigated the existence and qualitative profiles of traveling front of the nonlocal Fisher equation for the single species u:

$$u_t = u_{xx} + u\left(1 - \int_{-\infty}^{+\infty} g(x-y)u(y,t)dy\right), \quad x \in \mathbb{R}, \ t > 0, \tag{1.1}$$

where the kernel g satisfies $g \geq 0, g(x) = g(-x)$ and $\int_{-\infty}^{+\infty} g(x)dx = 1$, and the kernel is of the form $g(x) = \frac{1}{2}\lambda e^{-\lambda|x|}, \lambda > 0$. He showed for a general g that traveling fronts exist for Eq. (1.1) if the nonlocality is sufficiently weak. Ashwin et al. [13] studied an integro-differential equation based on the Fisher equation with a convolution term which introduces a time-delay in nonlinearity

$$u_t = u_{xx} + u - u(g * u), \quad x \in \mathbb{R}, \ t > 0, \tag{1.2}$$

where $g * u = \int_{-\infty}^{t}\int_{-\infty}^{+\infty} g(x-\xi, t-\tau)u(\xi,\tau)d\xi d\tau$, and the kernel is of the form $g(x,t) = \frac{b}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}e^{-bt}$, $b > 0$. They proved that the traveling wave front solutions persist when the delay in suitably small. Ai [15] considered the generalized Fisher equations with spatio-temporal delays of the form

$$u_t = u_{xx} + F((g_1 * \varphi_1)(t,x), \cdots, (g_m * \varphi_m)(t,x), \varepsilon), \quad x \in \mathbb{R}, \ t > 0,$$

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(1.3)

where $\varepsilon = (\mu, \varepsilon_1, \cdots, \varepsilon_m), \mu \in \mathbb{R}^{l_1} (l_1 \geq 0)$ is a small parameter, F is a $C^2(\mathbb{R}^{l+m})$ function with $l = l_1 + m$, for $j = 1, \cdots, m, \varphi_j \in C^2(\mathbb{R}^{l+1})$ and the convolutions is of the form $(g_j * \varphi_j)(t, x) = \int_0^{+\infty} \int_{-\infty}^{+\infty} g_j(s, y)\varphi_j(u(t-s, x-y), \varepsilon)dyds$, and each $g_j(t, x)$ takes one of the four special forms, for more details in [15]. The persistence of traveling wave fronts of generalized Fisher equations with very general kernels for all sufficiently small parameters are obtained by using geometric singular perturbation theory. The existence of the minimal wave speed, a continuum of wave fronts, and the global uniqueness of the physical wave front with each wave speed are also established. Jin [16] studied a generalized Fisher equation with nonlocal delay

$$\frac{\partial u}{\partial t} = \triangle u + u(1 - g * u^n), \quad x \in \mathbb{R}^N, \quad t > 0, \tag{1.4}$$

where $n \geq 1$, $g(x, t)$ is a nonnegative, integrable and even function with $\int_0^{+\infty} \int_{\mathbb{R}^N} g(x, t)dxdt = 1$ and $g * u^n(x, t) = \int_0^{+\infty} \int_{\mathbb{R}^N} g(x - y, t - s)u^n(y, s)dyds$, and the kernel g is taken to be $g(x, t) = \frac{b}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}} e^{-bt}, b > 0$. The existence of traveling wave fronts connecting two uniform steady states are obtained by using upper and lower solutions approach. The convergent rate of traveling wave solutions at infinity is also investigated.

In addition to the existence and asymptotic behavior of traveling wave solutions, the stability, uniqueness of traveling wave solutions in reaction-diffusion equations have been widely studied by many researchers, see [23–28] and the references therein. Smith and Zhao [23] first established the existence and comparison theorem of solutions in a quasimonotone reaction-diffusion bistable equation with a discrete delay by appealing to the theory of abstract functional differential equations, and the global asymptotic stability, Liapunov stability and uniqueness of traveling wave solutions are proved by the elementary sub- and supersolutions comparison and the squeezing technique developed by Chen [24], see also [26,27]. In fact, the earlier results concerning with this topic are due to Schaaf [3], who proved linearized stability for Fisher nonlinearity by a spectral method. Mei et al. [51] proved the stability of monotone traveling waves of Nicholson's blowflies equation with time delays by using weighted energy method, and further employed by many researchers to prove the stability of monotone traveling waves of various monostable reaction-diffusion equation with delays [52–55], and the references therein. Wu [56] extended the weighted energy method developed by Mei to prove the asymptotic stability of traveling waves for the delayed reaction-diffusion equations with crossing-monostability. It is worth mentioning that Ma and Zou [25] generalized the method of Chen and Guo [26,27] to a class of discrete reaction-diffusion monostable equation with delay and obtained the existence, uniqueness and stability of traveling wave fronts. Lv and Wang [26] also studied the existence, uniqueness and asymptotic behavior of traveling wave fronts for a vector disease model by the methods developed by Fenichel [32] and Smith and Zhao [23].

Consider here the following generalized Fisher equation with nonlocal delay

$$u_t = u_{xx} + pu(1 - f * u^r)(q + u^r), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{1.5}$$

where $u(x, t)$ is a scalar function which represents a population at time t and at position x, with $p > 0$, $q > 0$, $r > 0$ and the spatio-temporal convolution $f{*}u^r$ is defined by

$$(f * u^r)(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} f(x - y, t - s)u^r(y, s)dyds, \tag{1.6}$$

and $f(x, t)$ is a nonnegative, integrable and satisfies the usual normalization assumption, namely

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} f(x, t)dxdt = 1,$$

so that the kernel does not affect the spatially uniform steady states, which in this model will be $u \equiv 0$ and $u \equiv 1$. The kernel $f(x, t)$ is taken to be

$$f(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \frac{1}{\tau} e^{-\frac{t}{\tau}}, \quad \tau > 0, \tag{1.7}$$

which is called the weak generic delay kernel. Many authors have studied the weak generic delay kernel, see, for example, Gourley and Ruan [14], Lin and Li [32], Ashwin [13], Lv and Wang [26] etc.

Eq. (1.5) can be viewed as the generalized Fisher Eq. (1.8) with nonlocal term $f{*}u^r$ which accounts for a weighted average of the values u^r at all points in the domain. It also describes the situation of a nonlocal diffusion and a generalized competition in the study of pattern formation in a single species population [9]. In model (1.5), temporal delays represent for the fact that a resource, once consumed, takes time to recover. Thus this model includes the effect of past populations. Spatial averaging means that individuals are moving around (by diffusion) and have therefore not been at the same point in space at different times in their history. A detailed derivation is provided in Britton [9] for the use of the kernel (1.7) to account for this. Hence, this type of Eq. (1.5) is believed to be more realistic than nondelayed Eq. (1.8). Mathematically, the term $f{*}u^r$ is nonlocal and therefore can modify the nature of the solutions of the PDE (1.8) dramatically. To the best of our knowledge, model (1.5) has not been investigated yet.

In the case of $\tau \to 0$, which means the delay is absent, then Eq. (1.5) reduces to

$$u_t = u_{xx} + pu(1 - u^r)(q + u^r), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.8}$$

which is the so called generalized Fisher equation involved more competition in the populations [29,30]. Fan [30] studied the analytic solution of Eq. (1.8) by using a combination of nonlinear transformations and symbolic computations. The solution of Eq. (1.8) has the form

$$u(x, t) = \left\{ \frac{1}{2} - \frac{1}{2} \tanh \left[\frac{r}{2} \sqrt{\frac{p}{r+1}} \left(x - (1 + q + qr) \sqrt{\frac{p}{r+1}} t \right) \right] \right\}^{\frac{1}{r}}, \tag{1.9}$$

and represents a wave traveling with a speed $c = (1 + q + qr)\sqrt{\frac{p}{r+1}}$, in the positive x direction. When $p = q = 1$ and $r = \frac{1}{2}$, Eq. (1.8) reduces to the standard form of Fisher equation

$$u_t = u_{xx} + u(1 - u), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.10}$$

which was suggested by Fisher [31] as a deterministic version of a stochastic model for the spatial spread of a favored gene in the population, but the discovery, investigation and analysis of traveling waves in chemical reactions was first presented by Luther [49], Showalter and Tyson [50].

Note that equations of various types can be derived from Eq. (1.5) by taking different delay kernels. For example, when the kernel is taken to be $f(x, t) = \delta(x)\delta(t)$, where $\delta(x)$ is the Dirac delta function, Eq. (1.5) becomes Eq. (1.8). While taking $f(x, t) = \delta(x)\delta(t - \tau)$, Eq. (1.1) becomes

$$u_t = u_{xx} + pu(1 - u^r(x, t - \tau))(q + u^r), \quad x \in \mathbb{R}, \quad t > 0, \quad \tau > 0,$$

which gives an equation with a discrete, rather than distributed. Recently, the following three special kernels

$$f(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \frac{t}{\tau^2} e^{-\frac{t}{\tau}}, \quad f(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}, \quad f(t) = \frac{t}{\tau^2} e^{-\frac{t}{\tau}},$$