Chaos, Solitons and Fractals 103 (2017) 544–554

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Mathematical modelling and analysis of two-component system with Caputo fractional derivative order

Kolade M. Owolabi [a,b]

[a] Department of Mathematical Sciences, Federal University of Technology, PMB 704, Akure, Ondo State, Nigeria
[b] Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein 9300, South Africa

ARTICLE INFO

Article history:
Received 16 April 2017
Revised 10 July 2017
Accepted 13 July 2017

MSC:
34A34
35A05
35K57
65L05
65M06
93C10

Keywords:
Caputo derivative
Exponential integrator
Fractional reaction-diffusion
Hopf bifurcation
Oscillations
Predator-prey system
Stability analysis

ABSTRACT

A class of generic spatially extended fractional reaction-diffusion systems that modelled predator-prey interactions is considered. The first order time derivative is replaced with the Caputo fractional derivative of order $\gamma \in (0,\ 1)$. The local analysis where the equilibrium points and their stability behaviours are determined is based on the adoption of qualitative theory for dynamical systems ordinary differential equations. We derived conditions for Hopf bifurcation analytically. Most significantly, existence conditions for a unique stable limit cycle in the phase plane are determined analytically. Our analytical findings are in agreement with the numerical results presented in one and two dimensions. The system of fractional nonlinear reaction-diffusion equations has demonstrated the usefulness of understanding the dynamics of nonlinear phenomena.

1. Introduction

Over the years, the system of fractional reaction-diffusion equations have gained a lot of scholars attention in the to study nonlinear phenomena that occur in various application areas of science, engineering and technology [1–5,7,15,25,26]. Of particular interest the formation of nonlinear patterns in high dimensions [6,11,12,33,36]. The evolution of pattern formation can best be described by the fractional order system due to the fact that the fractional-order derivatives involved take into account the whole history of the equation is known as the memory effect [9,14,37].

A fractional reaction-diffusion system is considered as a generalisation of the classical reaction-diffusion system with the derivative of arbitrary (real) order. The fractional reaction-diffusion system is obtained by replacing the first-order time derivative order by γ defined on $0<\gamma<1$, or the second-order spatial derivative power by β, on the interval $1<\beta<2$. We can also replace both to obtain time-space fractional reaction-diffusion system,

$$\frac{\partial^{\gamma} U}{\partial t^{\gamma}} = d\frac{\partial^{\beta} U}{\partial x^{\beta}} + F(u), \tag{1.1}$$

where $\frac{\partial^{\gamma} U}{\partial t}$ and $\frac{\partial^{\beta} U}{\partial x^{\beta}}$ are the derivative operators, parameter d is the dimensionless diffusion coefficient, or the diagonal matrix $d=\mathrm{diag}[d_i^2]>0$, and the term $F(U)$ accounts for the local (or reaction) kinetics.

A time fractional reaction-diffusion version of (1.1) is a system of the form

$${}^{C}\mathcal{D}_t^{\gamma} U(t) = d\frac{\partial^2 U}{\partial x^2} + F(U) \tag{1.2}$$

with two variables $U=(u,v)^T$ on $x\in(0,\ L)$, subject to any of the boundary conditions (see, [30]): (i) For the infinite system, $x\in(-\infty,\infty)$, here $\mathbf{R}$ is a subset of $(-\infty,\infty)$. (ii) For $x\in[0,L]$, $\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(L,t)=0$, no-flux or Neumann boundary condition for a finite system, and (iii) $x\in[0,L]$, $\mathbf{u}(0,t)=\mathbf{u}(L,t)=\mathbf{u}_a$, called the Dirichlet or fixed concentration boundary condition, also for a fixed system, where $u(t,\ \mathbf{x})\in\mathbf{R}^n$, F: $\mathbf{R}^n\to\mathbf{R}$, with component

E-mail addresses: mkowolax@yahoo.com, kmowolabi@futa.edu.ng

http://dx.doi.org/10.1016/j.chaos.2017.07.013
0960-0779/

v having similar expression. And $F=(f,g)^T$, $f(u,\ v)$, $g(u,\ v)$ are smooth reaction kinetics.

The fractional derivative ${}^C\mathcal{D}_t^{\gamma}u(t)$ on the left-hand side of (1.2) is defined by the Caputo derivative of order $0<\gamma<1$ in time, and is given as [37]

$$ {}^C\mathcal{D}_t^{\gamma}u(t)=\frac{\partial^{\gamma}u}{\partial t^{\gamma}}=\frac{1}{\Gamma(n-\gamma)}\int_0^t\frac{u^{(n)}(\tau)}{(t-\tau)^{\gamma+1-n}}d\tau \tag{1.3}$$

where $n-1<\gamma<n$. The space fractional reaction-diffusion system is obtained by substituting $\gamma=1$ in (1.1). The first term to the right-hand side of (1.1) can be represented by the Riemann–Liouville fractional derivative, that is,

$$\frac{\partial^{\beta}u}{\partial x^{\beta}}=\frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial x}\int_0^x\frac{u(s,x)}{(x-s)^{\beta}}ds, \tag{1.4}$$

where $1<\beta\le 2$. It should be mentioned that when $\beta=2$, we recall the standard case.

The aim of this paper is structured into sections as follows. Analysis of the main result using the Mittag–Leffler and discretization techniques for fractional reaction-diffusion systems are presented in Section 2. Specific examples are analysed for stability (Hopf and Turing instabilities) in Section 3, to gain a full understanding of the parameters range when numerically simulating the whole systems. Numerical experiments in one and two dimensions are reported in Section 4. The conclusion will be drawn in Section 5.

2. Main results and discretization techniques

In thus section, we present the main results and the methods of discretization.

2.1. Main results

Let us consider the differential equations

$$ {}^C\mathcal{D}_a^{\gamma}u(t)=g(t)u+f(t),\ \ t_a<t<t_a+T,\ T>0,\ u(t_a)=u_a,\quad 0<\gamma<1 \tag{2.5}$$

where $g(t)$ and $f(t)$ are both continuous on closed interval $t_a<t<t_a+T$, and the term ${}^C\mathcal{D}_a^{\gamma}$ represents the Caputo fractional derivative of order γ. Throughout this work, we let $a=0$, and seek solution $u(t)$ of (2.5), which is assumed to be continuously differentiable, that is C^1 on $[t_0,t_0+T]$. Bear in mind that if $u\in C^1([t_0,t_0+T],\mathbb{R})$, then $u\in C_{\varrho}^1$ on $([t_0,t_0+T],\mathbb{R})$, see [18] for details.

The solution of (2.5) is also the solution of the equation of the form

$$u(t)=u_0+\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}g(\xi)u(\xi)d\xi+\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}f(\xi)d\xi \tag{2.6}$$

for $t_0<t<t_0+T, T>0$. Eq. (2.6) is often referred to as the Volterra fractional integral equation [18,37,38].

Theorem 2.1. *Let $g(t)$ and $f(t)\in C([t_0,t_0+T],\mathbb{R})$ then we can symbolically write the solution of (2.5) as*

$$u(t)=u_0e^{{}^C\mathcal{D}_0^{-\gamma}g(t)}+\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}e^{{}^C\mathcal{D}_0^{-\gamma}g(\xi)}f(\xi)d\xi,$$

for $t_0<t<t_0+T,\ T>0,\ \gamma\in[0,1]$.

Proof. Since (2.6) is considered as the solution of (2.5), we define the $u_n(t)$ by

$$u_n(t)=u_0+\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}g(\xi)u_{n-1}(\xi)d\xi+\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}f(\xi)d\xi,\ t_0<t<t_0+T,\ T>0. \tag{2.7}$$

Beginning with the initial approximation $u_0(t)=u_0$, we have

$$u_1(t)=u_0+\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}g(\xi)u_0d\xi+\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}f(\xi)d\xi,t_0<t<t_0+T,\ T>0 \tag{2.8}$$

which we simplify into

$$u_1(t)=u_0[1+\mathcal{D}_0^{-\gamma}g(t)]+\mathcal{D}_0^{-\gamma}f(t).$$

Since $g(t), f(t), \mathcal{D}_0^{-\gamma}g(t)$ and $\mathcal{D}_0^{-\gamma}f(t)$ are continuous on $t_0\le t\le t_0+T,\ T>0$, then they are uniformly continuous.

If we let $g(t)=\kappa$, (a constant), then

$$u_1(t)=u_0\left(1+\frac{\kappa(t-t_0)^{\gamma}}{\Gamma(\gamma+1)}\right)+\int_{t_0}^t\frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)}f(\xi)d\xi. \tag{2.9}$$

Again,if $|g(t)|\le\kappa$, then

$$|u_1(t)|\le|u_0|\left(1+\frac{\kappa(t-t_0)^{\gamma}}{\Gamma(\gamma+1)}\right)+\int_{t_0}^t\frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)}|f(\xi)|d\xi. \tag{2.10}$$

This shows that $u_1(t)$ is uniformly continuous on closed interval $t_0\le t\le t_0+T,\ T>0$. By following this process, we obtain

$$u_2(t)=u_0+\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}g(\xi)u_1d\xi+\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}f(\xi)d\xi, \tag{2.11}$$

which we simplify into

$$u_2(t)=u_0\{1+\mathcal{D}_0^{-\gamma}g(t)+\mathcal{D}_0^{-\gamma}(g(t)\mathcal{D}_0^{-\gamma}g(t))\}+\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}g(\xi)\mathcal{D}_0^{-\gamma}f(\xi)d\xi+\mathcal{D}_0^{-\gamma}f(t). \tag{2.12}$$

By interchanging the order of integration in the form

$$\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}g(\xi)\left\{\int_{t_0}^t(\xi-\omega)^{\gamma-1}f(\omega)d\omega\right\}d\xi=\frac{1}{\Gamma(\gamma)}\int_{t_0}^t(t-\xi)^{\gamma-1}f(\xi)\left\{\int_{t_0}^t(\xi-\omega)^{\gamma-1}g(\omega)d\omega\right\}d\xi, \tag{2.13}$$

we finally obtain

$$u_2(t)=u_0\{1+\mathcal{D}_0^{-\gamma}g(t)+\mathcal{D}^{-\gamma}(g(t)\mathcal{D}_0^{-\gamma}g(t))\}+\mathcal{D}_0^{-\gamma}\{(1+\mathcal{D}_0^{-\gamma}g(t))f(t)\}.$$

Also, if $|g(t)|\le\kappa$, then

$$|u_2(t)|\le|u_0|\left[1+\frac{\kappa(t-t_0)^{\gamma}}{\Gamma(\gamma+1)}+\frac{\kappa(t-t_0)^{2\gamma}}{\Gamma(2\gamma+1)}\right]+\int_{t_0}^t(t-\xi)^{\gamma-1}\left[\frac{1}{\Gamma(\gamma)}+\frac{\kappa(t-t_0)^{\gamma}}{\Gamma(2\gamma)}\right]|f(\xi)|d\xi \tag{2.14}$$

which shows that $u_2(t)$ is continuous on $[t_0,t_0+T]$. Since $u_1(t)$ and $u_2(t)$ are uniformly continuous on $[t_0,t_0+T],T>0$, by