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Numerical treatment of fractional order Cauchy reaction diffusion equations

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ABSTRACT

In this manuscript, an approximate method for the numerical solutions of fractional order Cauchy reaction diffusion equations is considered. The concerned method is known as optimal homotopy asymptotic method (OHAM). With the help of the mentioned method, we handle approximate solutions to the aforesaid equation. Some test problems are provided at which the adapted technique has been applied. The comparison between absolute and exact solution are also provided which reveals that the adapted method is highly accurate. For tabulation and plotting, we use matlab software.

1. Introduction

In the recent few decades the basic importance of fractional order partial differential equations(FPDEs) is well known fact in various physical and engineering disciplines. As most of the physical phenomenons can be modeled by using (FPDEs) in numerous fields of science and engineering like biology, chemistry, mechanics, economics, polymer, aerodynamics, biophysics, control theory and many more, (see for example [1,2,3,6]). It has been frequently observed in various physical phenomenons that the classical differential operator, being local in nature. It is very difficult task to form mathematical model to every physical phenomena via classical differential operator, because it can not well explain certain phenomenons specially problems having hereditary properties. On other hand fractional order-operators are nonlocal in nature and have memory effects as well as the embedded capability to explain and describe physical phenomenons which can not be described accurately via classical differential equations. The nonlocal property of fractional order-operators in the field of (FPDEs) is a reason of its popularity in modeling certain chemical, psychological, biological, physical, thermoplasticity and mechanical system . One of the nonlocal process is the diffusion processes and is nicely modeled by several researchers via fractional order diffusion equations. Among others, we want to bring the attention of the readers to [7], in which the authors studied the fractional diffusion in discreet case. In [8], the authors investigate fractional order diffusion and presents some simulations results. In [9], a detailed analysis is presented for the numerical investigation of distributed order diffusion equations. Gómez–Aguilar [10,11], presented a detailed study on the application of fractional order diffusion equations to some applied problems of mechanics. The fractional order operators are nonlocal operators and provide greater degree of freedom in the models as compare to classical integer order which is local operator and does not allow greater degree of freedom for modeling. On the other hand the computation complexities involved in fractional order models is a goal and difficulty to solve them. Some times, we are not capable to obtain the exact analytic solution of each and every (FPDEs). However there exists a large variety of schemes, which have been proved to be supportive in obtaining approximate solution of the fractional order problems. Among others, we want to bring the attention of the readers to these schemes like, Fourier series method(FSM), Finite difference

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method(FDM), Adomian decomposition method(ADM), Method of radial base function(MRBFs), Homotopy analysis method(HAM), Wavelet techniques(WT), Spectral methods [12,13] and many more. These schemes have there own merits and demerits. Some of them provide a very good approximation. For instance, see the articles [14–20]. Keeping in view , the above methods and the computation complexities involved in fractional order models, we are introducing a modified scheme known as optimal homotopy asymptotic method(OHAM) which is easy with respect to geometry and implementation to handle the computation complexities in constructing the approximate solution as well exact solution of the (FPDEs). This method was introduced by Marinca et al. [21], in 2008, but we have organized this method with little bit changes in its original scheme for solving different type of (FPDEs). We have testified this proposed method by considering the test problems that are fractional order partial reaction diffusion equation and its various cases. The fractional order reaction diffusion equation [4,5], is given by

$$\frac{\partial^{\beta}\omega(x,t)}{\partial t^{\beta}} = c\frac{\partial^{2}\omega(x,t)}{\partial x^{2}} + r(x,t)\omega(x,t),\ (x,t)\in\Omega. \tag{1}$$

If $\beta = 1$, then it becomes classical reaction diffusion equation and the term $c(x,t)\frac{\partial^2\omega(x,t)}{\partial x^2}$ is diffusion term and $r(x,\ t)\omega(x,\ t)$ represents the reaction term. where $\omega(x,\ t)$ is the concentration, $r(x,\ t)$ is the reaction parameter and c is the diffusion coefficient.

The plan of our work is organized as: In Section 2, we have provided some basic properties and definitions form fractional calculus. The basic idea (OHAM) has been presented in Section 3. In Section 4, we have tested various problems to support the accuracy and efficiency of the proposed method. The last section is devoted to a short conclusion.

2. Preliminaries

In this section, we recall some basic definitions and known results of fractional calculus and applied analysis, (see [1,2,3,7]).

Definition 2.1. The Rieman–Liouville fractional integral of order $\alpha\in\mathbb{R}_{+}$ of a function $h(t)\in L([0,1],\mathbb{R})$ is defined by

$$J_0^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds, \tag{2}$$

provided that the integral on the right side converges.

Definition 2.2. For $\mu\in\mathbb{R}$, a function $f:\mathbb{R}\to\mathbb{R}^{+}$ is said to be in the space C_{μ} if it can be written as $f(x) = x^{q}f_1(x)$ with $q>\mu, f_1(x)\in C[0,\ \infty)$ such that $f(x)\in C_{\mu}^{n}$ if $f^{(n)}\in C_{\mu}$ for $n\in\mathbb{N}\cup\{0\}$.

Definition 2.3. The Caputo fractional order derivative of a function $h\in C_{-1}^{n}$ with $n\in\mathbb{N}\cup\{0\}$ is defined by

$$D_t^{\alpha}h(t) = \begin{cases} J^{n-\alpha}f^{(n)}, & n-1<\alpha\le n,\ n\in\mathbb{N}, \\ \frac{d^n}{dt^n}h(t), & \alpha = n,\ n\in\mathbb{N}. \end{cases} \tag{3}$$

Note: Throughout this paper, we use fractional order derivative in Caputo sense.

Definition 2.4. A two parameter Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k\alpha+\beta)}. \tag{4}$$

From (4) for $\alpha = \beta = 1$, we get $E_{1,1}(t) = e^{t}$ and $E_{1,1}(-t) = e^{-t}$.

3. Theory of proposed method

In this section, we first recall some results from [22–25].

Consider the fractional order Cauchy reaction diffusion Eq. (1) as

$$\frac{\partial^{\beta}\omega(x,t)}{\partial t^{\beta}} = N(\omega(x,t)) + g(x,t) \quad 0<\beta\le 1,\ x\in\Omega,\ t\in[a_0,\ a_1], \tag{5}$$

corresponding to initial conditions

$$A\left(\omega(x,\ t),\ \frac{\partial\omega(x,t)}{\partial t}\right) = 0, \quad t\in\{a_0,\ a_1\}, \tag{6}$$

where, the operator $\frac{\partial^{\beta}}{\partial t^{\beta}}$ is described in the Caputo sense , the term $N(\omega(x,\ t))$ will be linear or non linear or both simultaneously, the function $\omega(x,\ t)$ is unknown solution which we will find, $g(x,\ t)$ is given known expression, x and t are spatial and temporal independent variables respectively, Ω is domain and A is boundary operator. According to the definition of optimal homotopy $\varpi(x,\ t,\ p)$: $\Omega\times[0,\ 1]\to R$ satisfies

$$(1-p)\left(\frac{\partial^{\beta}\varpi(x,t,p)}{\partial t^{\beta}} - g(x,t)\right) - H(t,p)\left(\frac{\partial^{\beta}\varpi(x,t,p)}{\partial t^{\beta}} - (N(\varpi(x,t,p)) + g(x,t))\right) = 0. \tag{7}$$

Where $p\in[0,\ 1]$ is auxiliary constant which is known as embedding parameter, $x\in\Omega$ and $H(t,\ p)$ is an arbitrary chosen auxiliary function. It is necessary that $H(t,\ p)$ must not equal to zero for all p except at $p = 0$. According to the defined homotopy

$$\varpi(x,t,p) = \omega_0(x,t) \text{ at } p = 0,$$
$$\varpi(x,t;p) = \omega(x,t) \text{ at } p = 1.$$

When $p\in[0,\ 1]$ varies in the defined homotopy ensures a speedy convergence of $\varpi(x,\ t,\ p)$ to the exact solution .The precise execution of the (OHAM), is purely based on the fair selection of the auxiliary function. The region of fast convergence of (OHAM) approximation to the exact solution depends strictly on auxiliary function $H(t,\ p)$. Essentially, the expression in auxiliary function follows the terms appearing in $N(\omega(x,\ t))$ such that the product of the auxiliary function and $N(\omega(x,\ t))$ to be of the similar form. As

$$H(t,p) = p\lambda_1(t,C_i) + p^2\lambda_2(t,C_i) + p^3\lambda_3(t,C_i) + \dots. \tag{8}$$

Where $C_i, i = 1,2,3,\dots$ are auxiliary constants and $\lambda_i(t,C_i)$, $i = 1,2,3,\dots$ is a function of t and C_i. But to choose $\lambda_i(t,\ C_i)$ is purely on the basis of terms appear in nonlinear part $N(\omega(x,\ t))$ and $g(x,\ t)$. On the subject of this valuable and logical point, we choose $\lambda_1(t,C_i) = C_1,\ \lambda_2(t,C_i) = C_2,\ \lambda_3(t,C_i) = C_3\ \dots$, for our chosen fractional problems in our work because the similar form of each solution of simpler problem has been obtained in the simulation section. Expanding $\varpi(x,\ t,\ p)$ in Taylor's series about p, we have

$$\varpi(x,t,p) = \omega_0(x,t) + \sum_{k=1}^{\infty}\omega_k(x,t)p^k,\ i = 1,2,3,\dots. \tag{9}$$

A key point is to be noted that Eq. (9) converges to the desire solution at $p = 1$ as

$$\tilde{\omega}(x,t) = \omega_0(x,t) + \sum_{k=1}^{\infty}\omega_k(x,t). \tag{10}$$

Generally speaking, someone may truncate Eq. (10) into finite terms for to obtain the solution in few iteration.

By substituting Eq. (9) into Eq. (7) and equating co-efficient of like powers of the p, we obtain zero-order, first-order, second-order and high order problems respectively as follows:

$$p^0:\ \frac{\partial^{\alpha}\omega_0(x,t)}{\partial t^{\alpha}} - g(x,t) = 0,$$