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Hopf bifurcation analysis of two zooplankton-phytoplankton model with two delays☆

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ABSTRACT

We take the delays due to gestation of two kinds of zooplankton as parameters, the dynamics of a two zooplankton-phytoplankton model is studied, we discussed the dynamics under six conditions: (1) $\tau_1 = \tau_2 = 0$,(2) $\tau_1 > 0, \tau_2 = 0$, (3) $\tau_1 = 0, \tau_2 > 0$, (4) $\tau_1 = \tau_2 > 0$, (5) $\tau_1 \in (0, \tau_{10})$, $\tau_2 > 0$, (6) $\tau_2 \in (0, \tau_{20})$, $\tau_1 > 0$, the Hopf bifurcation about condition (5) should be studied by center manifold theorem and normal form. At last, some simulations are given to support our results

1. Introduction

As we know, aquatic ecosystem is important for our environment, where phytoplankton contribute to our climate by absorbing much carbon dioxide from atmosphere and creating oxygen, besides they act as the basis of food chain for aquatic ecosystem, zooplankton exist nearly in all marine and limnic environment, they offer other aquatic animal of food, so the work about dynamics of phytoplankton-zooplankton should be valuable.

We know the density of plankton always increase and decrease or keep invariant for long time, why these situations occur? it is a interesting topic which cause many works, toxic phytoplankton are considered as factors, harmful algal bloom in [13] have adverse effect on the bloom of aquatic population, the phytoplankton produce toxin to preventing grazing of zooplankton, some works [14,15] describe the toxin play role on the bloom of plankton population, besides there exist many works about stability and bifurcation of phytoplankton-zooplankton system [7,16,17].

We know delays always exist in biology systems [18,19], the dynamics always be determined by relative delays [1,20–22], such as Hopf bifurcation occur when the delay spent by zooplankton migration in horizontal and vertical direction [1] cross the critical value.

In the field of delayed plankton system, two kinds of delays are considered always, there are the gestation delay and delay required for maturity of toxic-phytoplankton, for example, the global Hopf bifurcation is discussed when gestation delay exist [9], the author discuss the Hopf bifurcation when the delay required for the maturity of toxin-phytoplankton [2,7] exist, assuming the gestation delay equal to the delay required for the maturity of toxic-phytoplankton, the author study the stability and global Hopf bifurcation [5]. But by now, there is seldom work about three kinds of plankton with delays. In [7], Hopf bifurcation of a two harmful phytoplankton-zooplankton system with two delays is studied:

$$\begin{cases} \frac{dP_1}{dt} = r_1P_1(1-\frac{P_1}{K}) - \alpha_1P_1P_2 - \rho_1P_1Z, \\ \frac{dP_2}{dt} = r_2P_2(1-\frac{P_2}{K}) - \alpha_2P_1P_2 - \rho_2P_2Z, \\ \frac{dZ}{dt} = (r_1P_1 + r_2P_2)Z - dZ - \theta_1P_1(t-\tau_1)Z - \theta_2P_2(t-\tau_2)Z, \end{cases} \tag{1.1}$$

some simulations are given on 7 cases: (1) $\tau_1 = 0.6, \tau_2 = 0$,(2) $\tau_1 = 8.0, \tau_2 = 0$,(3) $\tau_1 = 1.5, \tau_2 = 0$,(4) $\tau_1 = 20, \tau_2 = 0$,(5) $\tau_1 = 6, \tau_2 = 5$,(6) $\tau_1 = 6, \tau_2 = 5.9$,(7) $\tau_1 = 15, \tau_2 = 6$, a two zooplankton-phytoplankton system with delay is studied in [2],

$$\begin{cases} \frac{dP}{dt} = rP(1-\frac{P}{K}) - \frac{\mu_1PZ_1}{\alpha_1+P} - \frac{\mu_2PZ_1}{\alpha_2+P}, \\ \frac{dZ_1}{dt} = \frac{\beta_1PZ_1}{\alpha_1+P} - \frac{\rho_1P(t-\tau)Z_1}{\alpha_1+P(t-\tau)} - d_1Z_1 - g_1Z_1^2, \\ \frac{dZ_2}{dt} = \frac{\beta_2PZ_1}{\alpha_2+P} - \frac{\rho_2P(t-\tau)Z_2}{\alpha_2+P(t-\tau)} - d_2Z_2 - g_2Z_2^2, \end{cases} \tag{1.2}$$

In real world, we know more than one delays always coexist, so there exist many works about the hopf bifurcation of two delays [3,4,6,8], the author analysis the bifurcation with two delays by center manifold and normal form, inspired by works above, similar to [2], we only consider the gestation delays of two zooplankton,

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system (1.2) became:

$$\begin{cases} \frac{dP}{dt} = rP(1-\frac{P}{K}) - \frac{\mu_1 PZ_1}{\alpha_1+P} - \frac{\mu_2 PZ_1}{\alpha_2+P}, \\ \frac{dZ_1}{dt} = \frac{\beta_1 P(t-\tau_1)Z_1(t-\tau_1)}{\alpha_1+P(t-\tau_1)} - \frac{\rho_1 P(t)Z_1}{\alpha_1+P(t)} - d_1Z_1 - g_1Z_1^2, \\ \frac{dZ_2}{dt} = \frac{\beta_2 P(t-\tau_2)Z_1(t-\tau_2)}{\alpha_2+P(t-\tau_2)} - \frac{\rho_2 P(t)Z_2}{\alpha_2+P} - d_2Z_2 - g_2Z_2^2, \end{cases} \tag{1.3}$$

where $P, Z_i, (i=1,2)$ denote the density of phytoplankton and zooplankton respectively, τ_1, τ_2 denote the gestation of the two zooplankton, r denote the growth rate of phytoplankton, K is the carrying capacity, μ_i is the ingestion rate for phytoplankton, β_i is ratio of biomass consumed for self-production, d_i is the mortality rate of zooplankton, ρ_i denote the rate of toxic substance produced by phytoplankton and g_i is the inhibitory effect of two competing zooplankton.

We know $\dot{Z}_i < 0$ when $\tau_i = 0$ and $\beta_i - \rho_i - d_i < 0$, we reject this situation by assuming $\beta_i - \rho_i - d_i > 0$ $(i = 1, 2)$, From [2], we obtain that equilibrium $E^*(P^*, Z_1^*, Z_2^*)$ exist if $h_3(\frac{d_1\alpha_1}{\beta_1-\rho_1-d_1}) > 0, h_3(\frac{d_2\alpha_2}{\beta_2-\rho_2-d_2}) > 0$, where

$$\begin{aligned} h_3(P) = {} & r(1-\frac{P}{K}(\alpha_1+P)^2(\alpha_2+P)^2 \\ & - \frac{\mu_1}{g_1}((\beta_1-\rho_1-d_1)P - d_1\alpha_1)(\alpha_2+P)^2 \\ & - \frac{\mu_2}{g_2}((\beta_2-\rho_2-d_2)P - d_2\alpha_2)(\alpha_1+P)^2, \end{aligned}$$

if the $h_3(P)$ is monotone for $P \in (P_0, P_1)$, the equilibrium E^* is unique, the definition of P_0, P_1 is showed in [2], all the discussion in this paper based on the existence and uniqueness of equilibrium E^*, we should demonstrate the positivity and boundness of the solution, which is essential in reality. The positivity means that the population of plankton will exist for long time and the boundness means that the population could not increase for ever, they are limited by many factors, we select Holling II functional response to represent the grazing of zooplankton on the toxic phytoplankton and analysis the dynamic of two zooplankton-phytoplankton with two delays about 6 cases: $(1)\tau_1 = \tau_2 = 0$,$(2)\tau_1 > 0, \tau_2 = 0$, $(3)\tau_1 = 0, \tau_2 > 0$, $(4)\tau_1 = \tau_2 > 0$, $(5)\tau_1 \in (0, \tau_{10})$, $\tau_2 > 0$, (6) $\tau_2 \in (0, \tau_{20})$, $\tau_1 > 0$, similar to [8], we discuss the Hopf bifurcation on case (5) by center manifold and normal form, all the analysis are demonstrated by simulations.

This paper is organized as follows: the positivity and boundness of solution is discussed in Section 2, the stability of positive equilibrium and existence of local Hopf bifurcation of system (1.3) is studied in Section 3, some simulations are given to support our result in Section 4, at last, we give conclusion in Section 5.

2. Positivity and boundness of solution

For system (1.3), we give the initial condition:

$$\begin{aligned} P(\theta) = \phi(\theta) \geq 0, Z_i(\theta) = \psi_i(\theta) \geq, \\ \theta \in [-\tau, 0], \phi(0) > 0, \psi_i(0) > 0, \end{aligned} \tag{2.1}$$

where $\tau = max(\tau_1, \tau_2)$, from the fundamental theory [10], system (1.3) with condition (2.1) admit uniqueness and existence of solution on $[0, +\infty)$, besides, we have

Lemma 2.1. *All the solution of system (1.3) with initial condition (2.1) are positive and bounded on* $[0, +\infty)$.

Proof. Let $(P(t), Z_1(t), Z_2(t))$ be a solution of system (1.3), we consider $Z_i(t)$ for $t \in [0, \tau]$.

$$\begin{aligned} \frac{dZ_i}{dt} & = \frac{\beta_i P(t-\tau_i)Z(t-\tau_i)}{\alpha + P(t-\tau_i)} - \frac{\rho_i P(t)Z_i(t)}{\alpha_i + P(t)} - d_iZ_i - g_iZ_i^2 \\ & \geq -\frac{\rho_i P(t)Z_i(t)}{\alpha_i + P(t)} - d_iZ - g_iZ_i^2, \end{aligned}$$

since $\phi(\theta) \geq 0$, $\psi_i(\theta) \geq 0$ for $\theta \in [-\tau, 0]$ we get

$$\begin{aligned} Z_i(t) \geq {} & \psi_i(0)exp\left(\int_0^t \left(-\frac{\rho_i P(s)}{\alpha_i + P(s)} - d_i - g_iZ_i(s)ds\right)\right) \\ & > 0, t \in [0, \tau] \end{aligned}$$

Thus, $Z_i(t)$ is positive for $t \in [0, \tau]$, similarly

$$\begin{aligned} P(t) = {} & \phi(0)exp\left(\int_0^t \left(r\left(1 - \frac{P(s)}{K}\right) - \frac{\mu_1 Z_1(s)}{\alpha_1 + P(s)} - \frac{\mu_2 Z_2(s)}{\alpha_2 + P(s)}\right)ds\right) \\ & > 0, t \in [0, \tau] \end{aligned}$$

so we could expand the result to $[\tau, 2\tau], \ldots, [n\tau, (n+1)\tau], n \in N$. Thus $P(t) > 0, Z_i(t) > 0, i = 1, 2$ for $t \geq 0$.

From the first equation of (1.3), we obtain $\dot{P} \leq rP(1-\frac{P}{K})$, so $\limsup_{t\to\infty} P(t) \leq K$, for $\varepsilon > 0$ sufficiently small, there exist sufficiently large $T > 0$ such that $P(t) < K + \varepsilon$ for all $t \geq T$, for $Z_i(t)$, $d \doteq \min(d_1, d_2)$, we define $W(t) = P(t-\tau_1) + P(t-\tau_2) + \frac{\mu_1}{\beta_1}Z_1(t) + \frac{\mu_2}{\beta_2}Z_2(t)$ for $t \geq 0$, Then

$$\begin{aligned} \dot{W} & = \frac{dP(t-\tau_1)}{dt} + \frac{dP(t-\tau_2)}{dt} + \frac{\mu_1}{\beta_1}\frac{dZ_1}{dt} + \frac{\mu_2}{\beta_2}\frac{dZ_2}{dt} \\ & \leq rP(t-\tau_1)\left(1 - \frac{P(t-\tau_1)}{K}\right) + rP(t-\tau_2)\left(1 - \frac{P(t-\tau_2)}{K}\right) \\ & \quad - \frac{d_1\mu_1}{\beta_1}Z_1 - \frac{d_2\mu_2}{\beta_2}Z_2 \\ & \leq -d_1\left(P(t-\tau_1) + \frac{\mu_1}{\beta_1}Z_1\right) + P(t-\tau_1)\left(d_1 + r - \frac{rP(t-\tau_1)}{K}\right) \\ & \quad - d_2\left(P(t-\tau_2) + \frac{\mu_2}{\beta_2}Z_2\right) + P(t-\tau_2)\left(d_2 + r - \frac{rP(t-\tau_2)}{K}\right) \\ & = -dW + \frac{K}{4r}((d_1+r)^2 + (d_2+r)^2), \end{aligned}$$

so by the comparison theory [12], we obtain $W(t) \leq W(0) + \frac{K((d_1+r)^2+(d_2+r)^2)}{4dr}$.

Thus we complete the proof. □

3. Hopf bifurcation

Before our discussion, transfer the equilibrium to origin by $x = P(t) - P^*, y = Z_1(t) - Z_1^*, z = Z_2(t) - Z_2^*$, system (1.3) became

$$\begin{cases} \frac{dx}{dt} = a_{11}x + a_{12} + a_{13}z + F_1, \\ \frac{dy}{dt} = a_{21}x + a_{22}y + a_{23}x(t-\tau_1) + a_{24}y(t-\tau_1) + F_2, \\ \frac{dz}{dt} = a_{31}x + a_{32}z + a_{33}x(t-\tau_2) + a_{34}z(t-\tau_2) + F_3. \end{cases} \tag{3.1}$$

where

$a_{11} = -\frac{rP^*}{K} + \frac{\mu_1 P^* Z_1^*}{(\alpha_1+P^*)^2} + \frac{\mu_2 P^* Z_2^*}{(\alpha_2+P^*)^2}$, $a_{12} = -\frac{\mu_1 P^*}{\alpha_1+P^*}$, $a_{13} = -\frac{\mu_2 P^*}{\alpha_2+P^*}$,
$a_{21} = -\frac{\rho_1\alpha_1 Z_1^*}{(\alpha_1+P^*)^2}$, $a_{22} = -\frac{\rho_1 P^*}{\alpha_1+P^*} - d_1 - 2g_1Z_1^*$, $a_{23} = \frac{\beta_1\alpha_1 Z_1^*}{(\alpha_1+P^*)^2}$,
$a_{24} = \frac{\beta_1 P^*}{\alpha_1+P^*}$, $a_{31} = -\frac{\rho_2\alpha_2 Z_2^*}{(\alpha_2+P^*)^2}$, $a_{32} = -\frac{\rho_2 P^*}{\alpha_2+P^*} - d_2 - 2g_2Z_2^*$,
$a_{33} = \frac{\beta_2\alpha_2 Z_2^*}{(\alpha_2+P^*)^2}$, $a_{34} = \frac{\beta_2 P^*}{\alpha_2+P^*}$,
$F_1 = g_1^* x^2 + g_2xy + g_3xz + g_4x^3 + g_5x^2y + g_6x^2z + hot$,
$F_2 = h_1x^2 + h_2y^2 + h_3x(t-\tau_1)^2 + h_4xy + h_5x(t-\tau_1)y(t-\tau_1) + h_6x^3 + h_7x(t-\tau_1)^3 + h_8x^2y + h_9x(t-\tau_1)^2y(t-\tau_1) + hot$,
$F_3 = k_1x^2 + k_2z^2 + k_3x(t-\tau_2)^2 + k_4xz + k_5x(t-\tau_2)z(t-\tau_2) + k_6x^3 + k_7x(t-\tau_2)^3 + k_8x^2z + k_9x(t-\tau_2)^2z(t-\tau_2) + hot$, where
$g_1^* = -\frac{r}{K} + \frac{\mu_1\alpha_1 Z_1^*}{(\alpha_1+P^*)^3} + \frac{\mu_2\alpha_2 Z_2^*}{(\alpha_2+P^*)^3}$, $g_2 = -\frac{\mu_1\alpha_1}{(\alpha_1+P^*)^2}$, $g_3 = -\frac{\mu_2\alpha_2}{(\alpha_2+P^*)^2}$,
$g_4 = -\frac{\mu_1\alpha_1 Z_1^*}{(\alpha_1+P^*)^4} - \frac{\mu_2\alpha_2 Z_2^*}{(\alpha_2+P^*)^4}$, $g_5 = \frac{\mu_1\alpha_1}{(\alpha_1+P^*)^3}$, $g_6 = \frac{\mu_2\alpha_2}{(\alpha_2+P^*)^3}$,
$h_1 = \frac{\rho_1\alpha_1 Z_1^*}{(\alpha_1+P^*)^3}$, $h_2 = -g_1$, $h_3 = -\frac{\alpha_1\beta_1 Z_1^*}{(\alpha_1+P^*)^3}$, $h_4 = -\frac{\alpha_1\rho_1}{(\alpha_1+P^*)^3}$, $h_5 = \frac{\beta_1\alpha_1}{(\alpha_1+P^*)^2}$,
$h_6 = -\frac{\alpha_1\rho_1 Z_1^*}{(\alpha_1+P^*)^4}$, $h_7 = \frac{\alpha_1\beta_1 Z_1^*}{(\alpha_1+P^*)^4}$, $h_8 = \frac{\alpha_1\rho_1}{(\alpha_1+P^*)^3}$, $h_9 = -\frac{\alpha_1\beta_1}{(\alpha_1+P^*)^3}$,