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Review

Dynamic behaviors of a fractional order two-species cooperative systems with harvesting☆

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ABSTRACT

This paper introduces a fractional order two-species cooperative systems with harvesting. By using the Routh-Hurwitz Conditions and the Lyapunov method, we provide several sufficient conditions to ensure the stability of the equilibriums for the system. Finally, a numerical example is presented to demonstrate the validity and feasibility of the theoretical result.

1. Introduction

Population models appearing in various fields of mathematical biology have been proposed and studied extensively with their universal and importance. Among them, mutualism which is the interaction of two species of organisms that benefits from each other plays an important role [1–9]. In [9], the author investegated an cooperative models developed to describe facultative mutualism as follows:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = r_1x_1\left[1 - \dfrac{x_1}{K_1} + b_{12}\dfrac{x_2}{K_1}\right] - e_1x_1, \\ \dfrac{dx_2(t)}{dt} = r_2x_2\left[1 - \dfrac{x_2}{K_2} + b_{21}\dfrac{x_1}{K_2}\right] - e_2x_2, \end{cases} \tag{1.1}$$

where r_i are the linear birth rates and the K_i are the carrying capacities which are all positive constants. The b_{12} and b_{21} measure the cooperative effect of x_2 on x_1 and x_1 on x_2 respectively which are all positive constants. The e_i are harvesting efforts on respective populations which are non-negative constants. In this paper the author showed two kinds of Lyapunov function to investigate the global stability for the system. Although a large amount of work has been done in studying dynamics on system (1.1), it has been restricted to integer order differential equations.

In recent years, many phenomena can be described successfully by fractional order differential equations. Fractional order differential equation generalizes the integer-order differential equation in which the order of derivatives can be any real or complex number. Because the conception of fraction-order may be more close to life than integer-order and allows greater degrees of freedom in the model, a large number of articles have been developed concerning the application of fractional order differential equations [10–24].

Motivated by the work above, in this paper we extend the system (1.1) to fractional order which becomes the following system:

$$\begin{cases} {}^cD_t^\alpha x_1(t) = r_1x_1\left[1 - \dfrac{x_1}{K_1} + b_{12}\dfrac{x_2}{K_1}\right] - e_1x_1, \\ {}^cD_t^\alpha x_2(t) = r_2x_2\left[1 - \dfrac{x_2}{K_2} + b_{21}\dfrac{x_1}{K_2}\right] - e_2x_2, \end{cases} \tag{1.2}$$

where all the parameters are as system (1.1).

The remainder of this paper is organized as follows. In Section 2, we present basic definitions and some known results. In Section 3, the local stability of equilibriums and uniform asymptotic stability of positive equilibriums are showed. In Section 4, a numerical example is provided to illustrate the effectiveness of the theoretical result. In the last section, a discussion of the paper is presented.

2. Preliminaries and definitions

There are some definitions for fractional derivative [14,15], Maybe the most used definition is Caputo definition owning to the advantage of Caputo approach that the initial conditions for fractional differential equations take on the same form as those for

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integer-order differentiation. In this paper, we also adopt the Caputo derivative.

Definition 2.1. The fractional integral of order $\alpha \in R_+$ of function $f(t)$ for $t > 0$ is defined as

$$I_t^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. The Caputo fractional derivative of order $\alpha \in (n-1, n], n \in N$ of $f(t)$ is defined as

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds. \tag{2.2}$$

Remark 2.1. When $0 < \alpha \le 1$ in (2.2), then the Caputo fractional derivative becomes

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds. \tag{2.3}$$

Throughout this paper, we always assume that $0 < \alpha \le 1$.

Lemma 2.3. *([11]) Consider the following commensurate fractional-order system.*

$$\frac{d^\alpha x}{dt^\alpha} = f(x), \quad x(0) = x_0, \tag{2.4}$$

with $0 < \alpha \le 1$ and $x \in R^n$. The equilibrium points of system (2.4) are calculated by solving the following equation: $f(x) = 0$. These points are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix $J = \frac{\partial f}{\partial x}$ evaluated at the equilibrium points satisfy: $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$.

Lemma 2.4. *(Uniform Asymptotic Stability Theorem [12]) Let $x = 0$ be an equilibrium point of system (2.4) and $D \subset R^n$ be a domain containing $x = 0$. Let $L(t, x)$: $[0, \infty) \times D \to R$ be a continuously differentiable function such that*

$$W_1(x) \le L(t, x(t)) \le W_2(x), \tag{2.5}$$

$${}^cD_t^\alpha L(t, x(t)) \le -W_3(x), \tag{2.6}$$

$\forall t \ge 0$, $\forall \in D$, $0 < \alpha \le 1$, where $W_1(x)$, $W_2(x)$ and $W_3(x)$ are continuous positive definite functions on D. Then $x = 0$ is uniformly asymptotically stable.

Remark 2.2. When $x = x^*$ is the equilibrium point of system (2.4) and satisfies the conditions of Lemma 2.4, then $x = x^*$ is uniformly asymptotically stable.

Proof. Let $x^* \neq 0$ be the equilibrium of system (2.4) and $y = x - x^*$. The αth order derivative of y is given by

$${}^cD_t^\alpha y = {}^c D_t^\alpha (x - x^*) = f(x) = f(y + x^*) \triangleq g(y),$$

where $g(0) = 0$ and in the new variable y, the system ${}^cD_t^\alpha y = g(y)$ has the equilibrium point at the origin. Therefore, from Lemma 2.4, we know that $y = 0$ is uniformly asymptotically stable, which means $x = x^*$ is uniformly asymptotically stable. □

Lemma 2.5. *([13]) Let $x(t) \in R_+$ be a continuous and derivable function. Then for any time instant $t \ge 0$*

$${}^cD_t^\alpha \left[x(t) - x^* - x^* \ln \frac{x(t)}{x^*} \right] \le \left(1 - \frac{x^*}{x(t)}\right) {}^c D_t^\alpha x(t), \tag{2.7}$$

where $x^* \in R_+, \alpha \in (0, 1]$.

3. Equilibrium points and stability

In order to evaluate the equilibrium points of system (1.2), let

$${}^cD_t^\alpha x_1(t) = 0, \qquad {}^cD_t^\alpha x_2(t) = 0,$$

that is

$$\begin{cases} r_1 x_1 \left[1 - \frac{x_1}{K_1} + b_{12}\frac{x_2}{K_1}\right] - e_1 x_1 = 0, \\ r_2 x_2 \left[1 - \frac{x_2}{K_2} + b_{21}\frac{x_1}{K_2}\right] - e_2 x_2 = 0. \end{cases} \tag{3.1}$$

We can obtain that system (1.2) has four equilibriums as follow:

- $E_0(0, 0)$ which is the trivial solution of system (1.2).
- $E_1(K_1A_1, 0)$ if $0 \le e_1 < r_1$;
- $E_2(0, K_2A_2)$ if $0 \le e_2 < r_2$;
- $E_3(x_1^*, x_2^*)$ if $b_{12}b_{21} < 1$, $0 \le e_1 < r_1$ and $0 \le e_2 < r_2$;

where $x_1^* = \frac{A_1K_1 + b_{12}A_2K_2}{1 - b_{12}b_{21}}, x_2^* = \frac{A_2K_2 + b_{21}A_1K_1}{1 - b_{12}b_{21}}, A_1 = 1 - \frac{e_1}{r_1}, A_2 = 1 - \frac{e_2}{r_2}$.

In the following section we evaluate the local asymptotical stability of the four equilibriums by Jacobian matrix. Firstly, let us discuss E_0.

Theorem 3.1. *If $r_1 < e_1$, $r_2 < e_2$, then the trivial solution E_0 of system (1.2) is locally asymptotically stable.*

Proof. The Jacobian matrix $J(E_0)$ for system (1.2) is

$$J(E_0) = \begin{pmatrix} r_1 - e_1 & 0 \\ 0 & r_2 - e_2 \end{pmatrix} \tag{3.2}$$

It is easy to know that with the conditions of Theorem 3.1 the eigenvalues corresponding to the equilibrium E_0 are

$$\lambda_1 = r_1 - e_1 < 0, \qquad \lambda_2 = r_2 - e_2 < 0.$$

which implied $|arg(\lambda_1)| = |arg(\lambda_2)| = \pi$. Hence by Lemma 2.3, we know that E_0 is locally asymptotically stable. □

Theorem 3.2. *If $e_1 < r_1$ and $r_2 < e_2$, then the equilibrium E_1 of system (1.2) is locally asymptotically stable. Similarly, if $r_1 < e_1$ and $e_2 < r_2$, the equilibrium E_2 of system (1.2) is locally asymptotically stable.*

Proof. We now discuss the locally asymptotic stability of E_1. The Jacobian matrix $J(E_1)$ is given as:

$$J(E_1) = \begin{pmatrix} r_1(1 - 2A_1) - e_1 & r_1 b_{12} A_1 \\ 0 & r_2\left(1 - \frac{b_{12}K_1A_1}{K_2}\right) - e_2 \end{pmatrix} \tag{3.3}$$

Hence the characteristic equation of $J(E_1)$ is

$$\begin{aligned} Q(\lambda) &= det(\lambda E - J(E_1)) \\ &= (\lambda - r_1(1 - 2A_1) + e_1)\left(\lambda - r_2\left(1 - \frac{b_{12}K_1A_1}{K_2}\right) + e_2\right) \\ &= 0 \end{aligned}$$

The eigenvalues corresponding to the equilibrium E_1 are

$$\lambda_3 = r_1(1 - 2A_1) - e_1 = e_1 - r_1$$

$$\lambda_4 = r_2\left(1 - \frac{b_{12}K_1A_1}{K_2}\right) - e_2$$

If $e_1 < r_1$, it is easy to know that $\lambda_3 < 0$ and $A_1 = 1 - \frac{e_1}{r_1} > 0$ which implies $\frac{b_{12}K_1A_1}{K_2} > 0$. Therefore, with the conditions $e_2 < r_2$, we can obtain $\lambda_4 = r_2(1 - \frac{b_{12}K_1A_1}{K_2}) - e_2 < r_2 - e_2 < 0$ which implies the equilibrium E_1 of the system is locally asymptotically stable. The similar result can be get about E_2. □