Chaos, Solitons and Fractals 92 (2016) 86–90

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Frontiers

Limit cycles in a class of switching system with a degenerate singular point

Feng Li*, Yuanyuan Liu

School of Science, Linyi University, Linyi 276005, Shandong, P.R. China

ARTICLE INFO

Article history:
Received 5 January 2016
Revised 14 August 2016
Accepted 12 September 2016

MSC:
34C05
34C07

Keywords:
Switching systems
Degenerate critical point
Lyapunov constant
Limit cycle
Center

ABSTRACT

Although switching systems have been investigated intensively, there are few results about limit cycles bifurcated from switching systems with degenerate singular point. In this paper, a method to compute focal values for degenerate critical point of switching systems was proposed. Furthermore, we studied a quartic system in order to illustrate the efficiency of our method.

1. Introduction

Hopf bifurcation has been investigated intensively because it is closely related to 16th problem of Hilbert. For planar ordinary differential equations, there were many good results for continuous systems. For example, one of the best-known results was $M(2) = 3$ [1] for a planar system with an elementary critical point. Here, $M(n)$ denotes the maximal number of small-amplitude limit cycles around a singular point with n being the degree of polynomials in the vector field. When $n = 3$, the authors constructed two different cubic systems to show there exist 9 limit cycles for cubic systems in [2] and [3]. Recently, Yu and Tian showed that there could be twelve limit cycles around a singular point in a planar cubic-degree polynomial system [4]. When a critical point is degenerate, its center problem has also been investigated by many authors, see [5–10]. There were also many results about the bifurcation of limit cycles [11–14], for more detail, see [15,16]. But for general system with a degenerate critical points it is still a hard work to solve its center problem and to determine the number of limit cycles. A special system with total degenerate critical point was investigated by Liu etc. in [17].

For non-smooth system, a quadratic switching system with nine limit cycles was constructed in [18] by Chen Xingwu et al. In [19] Llibre et al. studied the maximum number of limit cycles that bifurcate from the periodic solutions of the family of isochronous cubic polynomial centers. Llibre and Mereu [20] also studied the maximum number of limit cycles which can bifurcate from the periodic orbits of the isochronous centers of discontinuous quadratic polynomial differential systems. The number of limit cycles bifurcated from the periodic orbits was discussed in [21]. Recently, Tian et al. constructed a Bautin switching system with ten limit cycles in [22].

In this paper we are concerned with the appearance of one limit cycle from a degenerate singular point for switching bi-dimensional systems. This phenomenon can be considered as a kind of generalized Hopf bifurcation. In this paper we study the following class of discontinuous planar systems of ordinary differential equations

$$\begin{aligned}
\frac{dx}{dt} &= -y(x^2+y^2)^n + \sum_{k=2n+2}^{\infty} F_k^+(x,y), \\
\frac{dy}{dt} &= x(x^2+y^2)^n + \sum_{k=2n+2}^{\infty} G_k^+(x,y), \qquad y > 0, \\
\frac{dx}{dt} &= -y(x^2+y^2)^n + \sum_{k=2n+2}^{\infty} F_k^-(x,y), \\
\frac{dy}{dt} &= x(x^2+y^2)^n + \sum_{k=2n+2}^{\infty} G_k^-(x,y). \qquad y < 0.
\end{aligned} \tag{1.1}$$

* Corresponding author.
E-mail address: lf0539@126.com.cn (F. Li).

http://dx.doi.org/10.1016/j.chaos.2016.09.012
0960-0779/

The aim of this paper is to establish a method to compute Lyapunov constant at a degenerate singular point for switching bi-dimensional systems (1.1). With the help of Mathematica, the method could be used directly to compute Lyapunov constant for some systems with a degenerate singular point in many practical problems.

The rest of the paper is organized as follows. In the next section, a method to compute Lyapunov constant at a total degenerate critical point of switching system is given. As an example, a quartic switching system with a degenerate critical point is investigated in order to illustrate the efficiency of our method in Section 3.

2. Lyapunov constant of a switching system with a degenerate singular point

In this section, we will give a method to compute Lyapunov constant of a switching system with a degenerate singular point. Planar polynomial systems with degenerate critical points could be written as

$$\frac{dx}{dt} = \sum_{2n+1}^{\infty} X_k(x,y),$$
$$\frac{dy}{dt} = \sum_{2n+1}^{\infty} Y_k(x,y), \tag{2.1}$$

where n is a positive integer. When $xY_{2n+1}(x,y) - yX_{2n+1}(x,y)$ does not change its sign, the origin is a center or a focus. Generally, suppose that

$$xY_{2n+1}(x,y) - yX_{2n+1}(x,y) \geq d(x^2+y^2)^{n+1}.$$

System (2.1) can be become

$$\frac{dr}{dt} = r^{2n+1}\sum_{k=0}^{\infty}\varphi_{2n+2+k}(\theta)r^k,$$
$$\frac{d\theta}{dt} = r^{2n}\sum_{k=0}^{\infty}\psi_{2n+2+k}(\theta)r^k, \tag{2.2}$$

by transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{2.3}$$

where $\varphi_k(\theta)$, $\psi_k(\theta)$ are polynomials of $cos\theta$ and $sin\theta$, given by

$$\varphi_{2n+2}(\theta) = \cos\theta X_{2n+1}(\cos\theta, \sin\theta) + \sin\theta Y_{2n+1}(cos\theta, sin\theta),$$

$$\psi_{2n+2}(\theta) = \cos\theta Y_{2n+1}(\cos\theta, \sin\theta) - \sin\theta X_{2n+1}(\cos\theta, \sin\theta).$$

Eq. (2.2) is equivalent to

$$\frac{dr}{d\theta} = r\frac{\varphi_{2n+2} + \sum_{k=1}^{\infty}\varphi_{2n+2+k}(\theta)r^k}{\psi_{2n+2} + \sum_{k=1}^{\infty}\psi_{2n+2+k}(\theta)r^k} \tag{2.4}$$

which is a special case of equation

$$\frac{dr}{d\theta} = r\sum_{k=1}^{\infty}R_k(\theta)r^k, \tag{2.5}$$

where $R_k(\theta)$ is continuous and differentiable, and $R_k(\theta+\pi) = (-1)^k R_k(\theta)$.

By the methods of small parameters of Poincaré, the solutions of (2.5) could be written as

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} v_k(\theta)h^k$$

Where $v_1(0) = 1, v_k(0) = 0, \forall k \geq 2$.

Submitting above solution into (2.5),

$$v_1'(\theta) = R_0(\theta)v_1(\theta),$$
$$v_2'(\theta) = R_0(\theta)v_2(\theta) + R_1(\theta)v_1(\theta)^2,$$
$$\cdots\cdots,$$
$$v_m'(\theta) = R_0(\theta)\Omega_{1,m}(\theta) + R_1(\theta)\Omega_{2,m}(\theta) + \cdots + R_{m-1}(\theta)\Omega_{m,m}(\theta),$$
$$\cdots\cdots, \tag{2.6}$$

$v_k(\theta)$ could be get one by one.

$$v_1(\theta) = e^{\oint_0^{\vartheta} R_0(\varphi)d\varphi},$$
$$\cdots\cdots,$$
$$v_m(\theta) = v_1(\theta)\oint_0^{\vartheta}\frac{R_1(\varphi)\Omega_{2,m}(\varphi) + \cdots + R_{m-1}(\varphi)\Omega_{m,m}(\varphi)}{v_(\varphi)}d\varphi,$$
$$\cdots\cdots, \tag{2.7}$$

for more detail, see [8].

For system (2.1), it is easy to obtain $R_0(\theta) = \frac{\varphi_{2n+2}(\theta)}{\psi_{2n+2}(\theta)}$, but it is difficult for further study. For simplify, the following systems

$$\frac{dx}{dt} = (\delta x - y)(x^2+y^2)^n + \sum_{2n+2}^{\infty} X_k(x,y),$$
$$\frac{dy}{dt} = (x + \delta y)(x^2+y^2)^n + \sum_{2n+2}^{\infty} Y_k(x,y). \tag{2.8}$$

will be studied in this paper, and we get

$$\varphi_{2n+2}(\theta) = \delta, \psi_{2n+2}(\theta) = 1.$$

When $\delta = 0$, Eq. (2.4) yields that

$$\frac{dr}{d\theta} = r\frac{\sum_{k=1}^{\infty}\varphi_{2n+2+k}(\theta)r^k}{1 + \sum_{k=1}^{\infty}\psi_{2n+2+k}(\theta)r^k}. \tag{2.9}$$

Furthermore, the successive function could be defined as following

$$\Delta(h) = \tilde{r}(2\pi, h) - h.$$

For switching system with a degenerate singular point, the classical method could not be used, we must find some new methods to solve this problems. The expression of (1.1) in polar coordinates

$$(R^+(r,\theta), 1 + \Theta^+(r,\theta)) \quad \theta \in [0, \pi],$$
$$(R^-(r,\theta), 1 + \Theta^-(r,\theta)) \quad \theta \in [\pi, 2\pi]. \tag{2.10}$$

Similarly, we could define half-return map as in [23]. The Lemma 2.1 in [23] yields that we could compute the positive half-return map of

$$\frac{dx}{dt} = -y(x^2+y^2)^n + \sum_{k=2n+2}^{\infty} F_k^+(x,y),$$
$$\frac{dy}{dt} = x(x^2+y^2)^n + \sum_{k=2n+2}^{\infty} G_k^+(x,y), \tag{2.11}$$

and

$$\frac{dx}{dt} = -y(x^2+y^2)^n + \sum_{k=2n+2}^{\infty} F_k^-(x,y),$$
$$\frac{dy}{dt} = x(x^2+y^2)^n + \sum_{k=2n+2}^{\infty} G_k^-(x,y). \tag{2.12}$$

The method could be briefly introduced by following figures. First of all, for the upper phase of system,

$$\frac{dx}{dt} = -y(x^2+y^2)^n + \sum_{k=2n+2}^{\infty} F_k^+(x,y),$$