Chaos, Solitons and Fractals 99 (2017) 7–15

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Energy cycle and bound of Qi chaotic system

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ARTICLE INFO

Article history:
Received 16 August 2016
Revised 5 January 2017
Accepted 20 March 2017

Keywords:
Li-Poisson bracket
Dissipation
Kolmogorov system
Casimir function, Bound

ABSTRACT

The Qi chaotic system is transformed into a Kolmogorov-type system, thereby facilitating the analysis of energy exchange in its different forms. Regarding four forms of energy, the vector field of this chaotic system is decomposed into four forms of torque: inertial, internal, dissipative, and external. The rate of change of the Casimir function is equal to the exchange power between the dissipative energy and the supplied energy. The exchange power governs the orbital behavior and the cycling of energy. With the rate of change of Casimir function, a general bound and least upper bound of the Qi chaotic attractor are proposed. A detailed analysis with illustrations is conducted to uncover insights, in particular, cycling among the different types of energy for this chaotic attractor and key factors producing the different types of dynamic modes.

1. Introduction

In the past 20 years, some numerical chaotic models such as the Chen system [1], Lü system [2], Qi chaotic system [3], and some hyperchaotic systems [4,5] were generated. These systems were constructed via mathematics and simulations whereas the Lorenz system [6] was modeled on a physical process. A few studies have investigated the application of the Lorenz system in meteorology [7] and mechanics [8]. The main research focus of these numerical systems as well as the Lorenz system has been on their dynamic analysis. Topics usually include bound analysis [9], aperiodic solutions, sensitivity to initial conditions, bifurcation theory [10], circuit implementations, calculation of Lyapunov exponents [11,12], fractional order [13], chaos-based communication, proof of chaos existence, chaos control [14], and synchronization. However, these aspects of the research cannot explain the mechanism or reason for the production of dynamic modes, and also cannot interpret the physical analogues of state variables. To explore them, the mechanics of these numerical systems must be investigated. The lines of study include force analysis, physical analogue interpretation, energy transformation between internal energy, and supplied energy. Arnold [15] presented a Kolmogorov system describing a dissipative-forced dynamic system or hydrodynamic instability with a Hamiltonian function. Pasini and Pelino [16] gave a unified view of the Kolmogorov and Lorenz systems, thereby providing a force analysis of the Lorenz system. The recurrence of dynamics [17] and energy cycling [8] for the Lorenz system were also investigated with the understanding obtained from the Kolmogorov system.

Although the derivation of the Lorenz system is different from these numerical chaotic systems, they all have similar vector fields and chaotic dynamics. Therefore, from the mechanic's point of view, both types of systems must be governed by similar forces. The transformation of a numerical chaotic system into a Kolmogorov system can build a bridge between physical chaotic systems and numerical chaotic systems. Qi and Liang [18] transformed the Qi four-wing chaotic system to a Kolmogorov system, performed a force analysis and interpreted the state of chaos as angular momentum.

Therefore, the Hamiltonian function and the Kolmogorov system provide a starting point in studying the mechanism for these numerical chaotic systems. Furthermore, the Casimir function, like enstrophy or potential vorticity in a fluid dynamic context, is very useful in analyzing stability conditions and global description of a dynamical system. It represents a constant of the motion of the Hamiltonian system; moreover it defines a foliation of the phase space [8,19]. The energetics of the Lorenz system using the Casimir function has already been studied [20].

The study of energy cycles is important in mechanics, for there is instantaneous exchange among kinetic energy, potential energy, dissipative and supplied energy in physical processes. Each type of energy is represented by a type of force. When multiple energy exchanges operate frequently and substantially, the mechanism becomes complicated, and chaos may arise. Because each type of torque has a corresponding energy, an analysis of energy can

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http://dx.doi.org/10.1016/j.chaos.2017.03.044
0960-0779/© 2017 Published by Elsevier Ltd.

reflect the torque characteristics. So far, there has been no study regarding the energy cycle of numerical chaotic systems.

In this paper, the vector field of the Qi chaotic system is decomposed into inertial torque, internal torque, dissipative torque, and external torque. Correspondingly, kinetic energy, potential energy, dissipative energy, and external energy are identified in the system. The rate of change of the Casimir energy for a Qi chaotic attractor governs the dynamics of the system. The mechanism for different dynamic modes is revealed from the combination of different torques, which explains the physical phenomena and the energy cycles. The bound of the chaotic attractor is given through the extremal points of the Casimir function. Normally, it is difficult to find the bound of a chaotic attractor; even if there is an available method [9], the positive definite matrix is exceedingly difficult to solve in the equation for Lyapunov stability.

The rest of the paper is organized as follows: Section 2 describes the transformation between the Qi chaotic system and the Kolmogorov-type system. Section 3 presents the decomposition of the system's energy into its four forms and analyzes the cycling of energy using the Hamiltonian and Casimir functions; the bound of the chaotic attractor is also proposed. Section 4 analyzes the energy cycling among the different dynamic modes and uncovers the reason for chaos generation. Finally, a conclusion is made.

2. Transformation of the Qi chaotic system into Kolmogorov system

The Qi chaotic system is presented in the form [3]

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_2x_3, \\ \dot{x}_2 &= cx_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - bx_3. \end{aligned} \tag{1}$$

Here, $a,\ b,\ c \in R^+$ are constant parameters of the system.

To discover the physical analogue of the state variables and mechanics of the system, we introduce the Kolmogorov system and the Euler equation. Arnold [15] presented a Kolmogorov system describing dissipative-forced dynamical systems or hydro-dynamic instability, written in 3D form

$$\dot{\mathbf{x}} = \{\mathbf{x},\ H\} - \Lambda\mathbf{x} + \mathbf{f}, \tag{2}$$

where $\mathbf{x} = [x_1x_2x_3]^{\mathrm{T}}$, $\{,\}$ represents the algebraic structure of the kinetic energy part of the Hamiltonian function of a system, denoted by H, and the Lie–Poisson structure is defined as [21]

$$\{F,\ G\} = -\mathbf{x}\cdot(\nabla F \times \nabla G), \tag{3}$$

where $F, G \in C^\infty(\mathbf{g}^*)$, $\mathbf{g}$ is Lie algebra. The positive definite diagonal matrix Λ represents the dissipative force and the last term $\mathbf{f}$ represents the external force.

The Euler equation without external force for an incompressible fluid or a free rigid body gives a Hamiltonian description, which can be written as [21]

$$\begin{aligned} \dot{x}_1 &= (\Pi_3 - \Pi_2)x_2x_3, \\ \dot{x}_2 &= (\Pi_1 - \Pi_3)x_1x_3, \\ \dot{x}_3 &= (\Pi_2 - \Pi_1)x_1x_2, \end{aligned} \tag{4}$$

where $\Pi_i = I_i^{-1}$, I_i is the principle moment of inertia for the group SO(3), and x_i is the angular momentum satisfying

$$x_i = I_i\omega_i, \tag{5}$$

with ω_i the angular velocity. Eq. (4) can be written in the succinct form

$$\dot{\mathbf{x}} = \mathbf{x} \times \mathbf{\Pi x} = \mathbf{x} \times \mathbf{\Omega}, \tag{6}$$

where $\mathbf{x} = [x_1\ x_2\ x_3]^{\mathrm{T}}$, $\mathbf{\Omega} = [\omega_1\ \omega_2\ \omega_3]^{\mathrm{T}}$, $\mathbf{\Pi} = diag(\Pi_1\ \Pi_2\ \Pi_3)$. Decomposing the Hamiltonian

$$H = K + U \tag{7}$$

with $U = 0$ and

$$K = \frac{1}{2}\left(\Pi_1x_1^2 + \Pi_2x_2^2 + \Pi_3x_3^2\right), \tag{8}$$

and replacing F by $\mathbf{x}$ and G by H in Eq. (3), Eq. (4) is equivalent to [21]

$$\dot{F} = \{F,\ H\}, \tag{9}$$

i.e.,

$$\dot{\mathbf{x}} = \{\mathbf{x},\ H\} = \mathbf{x} \times \mathbf{\Pi x}. \tag{10}$$

We find that under a pure inertial force, the Kolmogorov system (2) is the same as the Euler Eq. (10).

Remark 1.

(1) The force (or torque) $\{\mathbf{x}, H\}$ in the Euler equation for a free rigid body is in the form of a fictitious force, either the inertial force or the centrifugal force, which consists of quadratic terms.
(2) The Kolmogorov system is a generalized Euler equation with dissipative and external forces.
(3) The quadratic terms are skew-symmetric, i.e., the sum of the coefficients of all quadratic terms (inertial force) in the Lie–Poisson bracket, [Eq. (3) or (10)], is zero.
(4) The Hamiltonian function H in the bracket of the Kolmogorov system only contains the kinetic energy term K, i.e., the potential energy vanishes ($U = 0$).

We now establish an analogy between the Qi chaotic system and the Kolmogorov system. Note that the sum of coefficients of all quadratic terms is nonzero in the Qi chaotic system. To satisfy the condition, we introduce the following transformation

$$y_1 = \alpha x_1,\ y_2 = x_2,\ y_3 = \beta x_3, \tag{11}$$

with inverse

$$x_1 = \frac{1}{\alpha}y_1,\ x_2 = y_2,\ x_3 = \frac{1}{\beta}y_3, \tag{12}$$

where α and β are nonzero constants. Hence, Eq. (1) is transformed into

$$\begin{aligned} \dot{y}_1 &= \frac{\alpha}{\beta}y_2y_3 - ay_1 + \alpha a y_2, \\ \dot{y}_2 &= -\frac{1}{\alpha\beta}y_1y_3 + \frac{c}{\alpha}y_1 - y_2, \\ \dot{y}_3 &= \frac{\beta}{\alpha}y_1y_2 - by_3. \end{aligned} \tag{13}$$

We choose parameters α and β such that

$$\frac{\alpha}{\beta} - \frac{1}{\alpha\beta} + \frac{\beta}{\alpha} = \frac{1}{\alpha\beta}(\alpha^2 + \beta^2 - 1) = 0. \tag{14}$$

to satisfy the skew-symmetric requirement of the Lie–Poisson bracket. To determine the potential energy from the Hamiltonian function, we make a further transformation

$$z_1 = y_1,\ z_2 = y_2,\ z_3 = y_3 - \gamma, \tag{15}$$

with inverse

$$y_1 = z_1,\ y_2 = z_2,\ y_3 = z_3 + \gamma. \tag{16}$$

Eq. (13) is transformed into

$$\begin{aligned} z_1 &= \frac{\alpha}{\beta}z_2z_3 - az_1 + \left(\alpha a + \frac{\alpha\gamma}{\beta}\right)z_2, \\ \dot{z}_2 &= -\frac{1}{\alpha\beta}z_1z_3 + \left(\frac{c}{\alpha} - \frac{\gamma}{\alpha\beta}\right)z_1 - z_2, \\ \dot{z}_3 &= \frac{\beta}{\alpha}z_1z_2 - bz_3 - b\gamma. \end{aligned} \tag{17}$$