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Long-time behavior of solutions and chaos in reaction-diffusion equations

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ABSTRACT

It is shown that members of a class (of current interest with many applications) of non-dissipative reaction-diffusion partial differential equations with local nonlinearity can have an infinite number of different unstable solutions traveling along an axis of the space variable with varying speeds, traveling impulses and also an infinite number of different states of spatio-temporal (diffusion) chaos. These solutions are generated by cascades of bifurcations governed by the corresponding steady states. The behavior of these solutions is analyzed in detail and, as an example, it is explained how space-time chaos can arise. Results of the same type are also obtained in the case of a nonlocal nonlinearity.

1. Introduction

Nonlinear reaction-diffusion equations are parabolic, semilinear partial differential equations (in one or more space variables) that have proven to be useful models for a wide variety of important applications. Notable examples include the Fisher and Fisher–Kolmogorov equations for growth-diffusion phenomena occurring in (genetic) population dynamics, the Hodgkin–Huxley and FitzHugh–Nagumo equations for modeling neuron spiking, the Belousov–Zhabotinsky chemical reactions, modeling Rayleigh–Benard convection, and the Zeldovich equation for modeling combustion phenomena. Moreover, reaction-diffusion equations are often used as a prototype for pattern formation and are considered by many experts to be an essential basis for biological morphogenesis. For further information on applications, the reader is referred to such sources as [5,14,23].

There is great interest in nonlinear reaction-diffusion equations from a purely mathematical perspective as well. The equations comprise an infinite-dimensional dynamical systems that exhibit an amazing spectrum of solution phenomena including periodic traveling waves, dissipative solitons, spiral waves, target patterns, bifurcation cascades, chaos and long-time dynamical configurations of great complexity. Of particular interest is characterization of the long-time behavior of the solutions, which concerns the dynamical regimes that the system may settle into as the time $t \to \infty$. If the system is dissipative, there is typically a global strange attractor, but things are especially challenging mathematically when the system is non-dissipative inasmuch as the long-time dynamics can be much more diverse and complicated than in the dissipative case. Considerable progress has been made in non-dissipative long-time reaction-diffusion dynamics, but many open problems remain. In this paper, we obtain new results on the long-time dynamics of two classes of non-dissipative nonlinear reaction-diffusion equations, used for mathematical models of many interesting phenomena, which both extend and provide

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additional mathematical details, especially regarding chaotic transitions, for systems analyzed in the literature.

We consider classes of reaction-diffusion equations with nonlocal and local nonlinearities; namely,the following problems

$$\frac{\partial u}{\partial t} - \Delta u + g(t, x, u) = 0, \quad (t, x) \in (0, T) \times \Omega, \tag{1.1}$$

$$u(0, x) = u_0(x) \in W^{1,2}(\Omega), \quad x \in \Omega, \quad T > 0 \tag{1.2}$$

$$u\Big|_{[0,T)\times\partial\Omega} = 0, \quad \Omega \subseteq \mathbb{R}^n, \ n \geq 1, \quad \partial\Omega \in Lip \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open domain, the boundary $\partial\Omega$ is Lipschitz, $g : L^{p_1}((0, T) \times \Omega) \longrightarrow L^{p_2}((0, T) \times \Omega)$ is a nonlinear operator and p_1, $p_2 > 1$ are fixed numbers. We assume that $g(t, x, u)$ is represented in one of the following forms:

$$\begin{aligned} &(\alpha): \ g(t, x, u) := a\|u\|_2^{\rho} u + h(t, x), \quad \text{or} \\ &(\beta): \ g(t, x, u) := a(t, x)|u|^{\rho} u + h(t, x), \end{aligned} \tag{1.4}$$

where $\|\cdot\|_2$ denotes the norm in $L^2(\Omega)$, $h(t, x)$ and $u_0(x)$ are given functions and $\rho > 0$, $a > 0$ are given constants. The problem posed above is investigated for the both cases separately: in the case of a nonlocal nonlinearity (i.e. $(4(\alpha))$), and in the case of a local nonlinearity (i.e. $(4(\beta))$).

We shall show that the partial differential equation (1.1) in the case of the nonlocal nonlinearity (i.e. the case $(4(\alpha))$) can possess an infinite number of different unstable solutions. In the local case $(4(\beta))$, which differs markedly from the nonlocal case $(4(\alpha))$, the problem allows an infinite number of different both unstable solutions, traveling along the space axis with arbitrary speeds, and traveling impulses, as well as an infinite number of different spatio-temporal (diffusion) chaotic states. These solutions are generated by cascades of static bifurcations of the evolution equation, which were studied, in particular, in [22]. As Ya. Sinai asserts in [21] "... the future of the chaos theory will be connected with new phenomena in nonlinear PDEs and other infinite-dimensional dynamical systems, where we can encounter absolutely unexpected phenomena".

The dynamics becomes much more complicated in the case of dynamical systems generated by partial differential equations (PDEs) largely due to the formation of spatially chaotic patterns. More generally, such systems may display interactions between spatially and temporally chaotic modes. One of the most challenging problems in this field is that of turbulence which displays statistical behavior in temporal and spatial regimes, whose correlations decay with distance in space and time, see e.g.[10,16,29]. It should be pointed out that there have been many investigations on this and related topics (see, for example, [1–4,6–9,12,17–19,21,29,30] and the references therein).

In what follows we study the Cauchy problem for an equation of the non-dissipative reaction-diffusion equation with an infinite-dimensional solution space; in particular, the corresponding steady-state problem has also infinitely many solutions. We show that the trajectories of solutions in the phase space depend on the choice of starting point on a sphere of the initial values. To be more precise, the initial value determines the long-time solution behavior, depending on the related Lyapunov exponent of the trajectory in phase space. The choice of starting point allows to determine where the solution of the problem will end up. If the limiting set is not one-dimensional, more complications can arise, including even the existence of absorbing manifolds. Moreover, if such an absorbing manifolds exists, its associated dynamics tends to be chaotic. We study this type of dynamic behavior and explain how space-time chaos can arise.

2. Existence in the autonomous case

2.1. The nonhomogeneous case

We begin by studying the problem in the case $(4(\alpha))$ when $g(t, x, u) := a\|u\|_2^{\rho} u + h(x)$, i.e. we consider the problem

$$\frac{\partial u}{\partial t} - \Delta u - a\|u\|_H^{\rho} u = h(x), \quad (t, x) \in (0, T) \times \Omega, \tag{2.1}$$

$$u(0, x) = u_0(x) \in W_0^{1,2}(\Omega) := H_0^1(\Omega), \quad u\Big|_{[0,T)\times\partial\Omega} = 0. \tag{2.2}$$

From (2.1) we compute that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 - a\|u(t)\|_2^{\rho+2} = \langle h, u\rangle, \qquad \|u(0)\|_2^2 = \|u_0\|_2^2 \tag{2.3}$$

which entails the inequalities

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_2^2 &\leq -\|\nabla u(t)\|_2^2 + 2a\|u(t)\|_2^{\rho+2} + \ \|h\|_{H^{-1}}^2 \\ &\leq -\lambda_1 ||u||_2^2) + 2a\|u(t)\|_2^{\rho+2} + \|h\|_{H^{-1}}^2, \end{aligned}$$

where $\|u(t)\|_{H_0^1} := \|\nabla u(t)\|_2$ and $\lambda_1 > 0$ is the first eigenvalue of the Laplace operator $-\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$.

Now we consider the solvability of this problem, which will be analyzed making use of the general results from [24]. We take $u_0 \in B_{r_0}^{H_0^1}(0)$, where $r_0 < \lambda_1$, and study the operator A generated by the problem: it acts, by definition, from $X := W^{1,2}(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap \{u(t, x)|u(0, x) = .u_0\}$ to $L^2(0, T; H^{-1}(\Omega))$. Next, we study the image of this operator A on the ball $B_r^X(0)$ for $r \in (0, r_0)$; more precisely, we define a subset M of the space $L^2(0, T; H^{-1}(\Omega))$ and a number $r \in (0, r_0)$, such that $A(B_r^X(0)) \subseteq M \subset L^2(0, T; H^{-1}(\Omega))$. In other words, we shall show that the problem is solvable in $B_{r_0}^X(0)$ for any $(h, u_0) \in M \times B_{r_0}^{H_0^1}(0)$. A detailed investigation requires some preliminary estimates.

So, let $u_0 \in B_{r_0}^{H_0^1}(0)$ for some number $r_0 < \lambda_1$; then we obtain

$$\frac{d}{dt}\|u(t)\|_2^2 \leq -\lambda_1\|u(t)\|_2^2 + 2a\|u(t)\|_2^{\rho+2} + \|h\|_{H^{-1}}^2.$$

Consequently, it is enough to study the following initial problem

$$d(y + c_1)/dt + \lambda_1(y + c_1) \leq 2a(y + c_1)^{\rho_1+1}, \quad y(0) = \|u_0\|_{H_0^1}^2, \tag{2.4}$$

where $y(t) := \|u(t)\|_2^2$, $\rho_1 = \frac{\rho}{2}$. Assume that the constant c_1 is chosen in such a way that the inequality

$$2a(y + c_1)^{\rho_1+1} - \lambda_1 c_1 \geq \|h\|_{H^{-1}}^2 + 2ay^{\rho_1+1}$$

holds.

Whence, one finds that

$$z' - \lambda_1\rho_1 z \geq -a\rho, \ z = (y + c_1)^{-\rho_1}, \ z(0) = \left(\|u_0\|_2^2 + c_1\right)^{-\rho_1},$$

which gives

$$(y + c_1)^{-\rho_1} \geq \left(\|u_0\|_2^2 + c_1\right)^{-\rho_1} e^{\lambda_1\rho_1 t} + \frac{2a}{\lambda_1} - \frac{2a}{\lambda_1}e^{\lambda_1\rho_1 t}$$

implying that

$$\begin{aligned} &\|u(t)\|_2^2 + c_1 \leq e^{-\lambda_1 t}\left(\|u_0\|_2^2 + c_1\right) \\ &\times \left[1 - \frac{2a}{\lambda_1}\left(\|u_0\|_2^2 + c_1\right)^{\rho_1}\left(1 - e^{-\lambda_1\rho_1 t}\right)\right]^{\rho_1^{-1}}. \end{aligned} \tag{2.5}$$

Therefore, we see from (2.5) that the functions u_0 and h should be selected from balls of respective spaces so as to satisfy the inequality

$$1 - \frac{2a}{\lambda_1}\left(\|u_0\|_2^2 + c_1\right)^{\rho_1} > 0 \Longrightarrow \|u_0\|_2^2 + c_1 < \left|\frac{\lambda_1}{2a}\right|^{\frac{2}{\rho}}. \tag{2.6}$$