Chaos, Solitons and Fractals 99 (2017) 109–115

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Analysis of an El Nino-Southern Oscillation model with a new fractional derivative

Jagdev Singh [a,*], Devendra Kumar [a], Juan J. Nieto [b]

[a] Department of Mathematics, JECRC University, Jaipur-303905, Rajasthan, India

[b] Departamento de Análise Matemática, Facultade de Matemáticas, Universidade de Santiago de Compostela, Santiago de Compostela 15782, Spain

ARTICLE INFO

Article history:
Received 24 July 2016
Revised 25 December 2016
Accepted 27 March 2017

Keywords:
Atmosphere-ocean
El Nino-Southern Oscillation
Caputo–Fabrizio fractional derivative
Fixed point theorem
Uniqueness

ABSTRACT

In this article, we analyze the El Nino–Southern Oscillation (ENSO) model in the global climate with a new fractional derivative recently proposed by Caputo and Fabrizio. We obtain the solution by using the iterative method. By using the fixed-point theorem the existence of the solution is discussed. A deeply analysis of the uniqueness of the solution is also discussed. And to observe the effect of the fractional order we presented some numerical simulations.

1. Introduction

In the latest years, El Niño/La Niña-Southern Oscillation (ENSO) is a quasiperiodic climate pattern that arises across the tropical Pacific Ocean every five years which has seized more and more observation of researchers due to its huge devastation. It is combined with two phases, the warm oceanic phase, El Niño, and the cold phase, La Niña. Hence, the nonlinear ENSO models are very essential and absorbing subject to analyze ocean climate, atmospheric physics and dynamical systems; see more in [1-15].

Fractional calculus has been attaining great admiration and significance due largely to its manifest applications in countless apparently various and universal fields of science and engineering. Fractional derivatives and fractional integrals are crucial subject matter in the branch of fractional calculus. Recently, many researchers and scientists around the globe studies in this special branch [16–25]. Caputo [18] investigated linear models of dissipation whose Q is almost frequency independent. Baleanu et al. [19] presented new trends in nanotechnology and fractional calculus applications. Kilbas et al. [20] reported theory and applications of fractional differential equations. Bulut et al. [21] reported the analytical study of differential equations of arbitrary order. Atangana and Alkahtani [22] analyzed the Keller-Segel model with a fractional derivative without singular kernel. Atangana and Alkahtani [23] studied non-homogenous heat model. Singh et al. [24] studied a fractional biological populations model. Kumar et al. [25] reported solutions of fractional reaction-diffusion equations. Choudhary et al. [26] obtained the analytic solution of the fractional order differential equations occurring in fluid dynamics.

Recently, Caputo and Fabrizio [16] introduced a new fractional derivative without singular kernel and in addition Losada and Nieto [17] studied the further properties. In this paper, we apply the new fractional derivative to the nonlinear ENSO model. The principal contribution of this work is determining the new fractional derivative to the nonlinear ENSO model and imparting in detail the exactness and uniqueness of the solution of the nonlinear model by using the fixed-point theorem. The formation of this paper is as follows: In Section 2, the Caputo–Fabrizio fractional order derivative is initiated. In Section 3, the ENSO model and approximate solution pertaining to the new Caputo–Fabrizio fractional derivative is discussed. In Section 4, by applying the fixed-point theorem the existence of coupled solutions is verified. In Section 5, the uniqueness of the coupled solutions is proved. In Section 6, the approximate solution of the problems is obtained. Results and discussion is reported in Section 7. And finally in the Section 8, the conclusions are discussed.

* Corresponding author.
E-mail addresses: jagdevsinghrathore@gmail.com (J. Singh), devendra.maths@gmail.com (D. Kumar), juanjose.nieto.roig@usc.es (J.J. Nieto).

http://dx.doi.org/10.1016/j.chaos.2017.03.058
0960-0779/

2. The Caputo–Fabrizio fractional order derivative

Definition 2.1. Let $u \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1]$ then the new Caputo fractional derivative is defined by

$$D_t^{\alpha}(u(t)) = \frac{M(\alpha)}{1-\alpha}\int_a^t u'(x)\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx, \tag{1}$$

where $M(\alpha)$ is a normalization of the function satisfies the condition $M(0) = M(1) = 1$ [16].

But, if $u \notin H^1(a, b)$ then, the derivative can be defined as:

$$D_t^{\alpha}(u(t)) = \frac{\alpha M(\alpha)}{1-\alpha}\int_a^t (u(t)-u(x))\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx. \tag{2}$$

Remark 1. If $\sigma = \frac{1-\alpha}{\alpha} \in [0, \infty), \alpha = \frac{1}{1+\sigma} \in [0, 1]$, then Eq. (2) presume the form

$$D_t^{\alpha}(u(t)) = \frac{N(\sigma)}{\sigma}\int_a^t u'(x)\exp\left[-\frac{t-x}{\sigma}\right]dx, N(0) = N(\infty) = 1. \tag{3}$$

$$\text{Moreover, } \lim_{\sigma\to 0}\frac{1}{\sigma}\exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t) \tag{4}$$

The corresponding anti-derivative resulted to be essential [17].

Definition 2.2. Let $0 < \alpha < 1$. If u is a function then the fractional integral of order α is defined by

$$I_{\alpha}^{t}(u(t)) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(s)ds, t \geq 0. \tag{5}$$

Remark 2. The remainder arising in the above definition of the fractional integral of Caputo type of the function of order $0 < \alpha < 1$ is a mean between u and its integral order one. This therefore requires,

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1, \tag{6}$$

where $M(\alpha) = \frac{2}{2-\alpha}, 0 \leq \alpha \leq 1$.

Further, Nieto and Losada [17] propounded the new Caputo derivative of order $0 < \alpha < 1$ can be reformulated as given by

$$D_t^{\alpha}(u(t)) = \frac{1}{1-\alpha}\int_a^t u'(x)\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx. \tag{7}$$

3. ENSO model pertaining to the new Caputo–Fabrizio fractional derivative and approximate solution

The coupled dynamical ENSO system was observed to recount the oscillating physical mechanism as given by [1-4]

$$\begin{aligned} \frac{dH(t)}{dt} &= cH(t) + \eta h(t) - \varepsilon H^3(t), \\ \frac{dh(t)}{dt} &= -\theta H(t) - \gamma h(t), \end{aligned} \tag{8}$$

with the initial conditions

$$H(0) = 1, h(0) = 1, \tag{9}$$

where c, η, γ, θ and ε are physical constant and ε is perturbation coefficient which is defined $0 < \varepsilon < 1$ and small ample. $H \in \Re$ is temperature of the eastern equatorial Pacific sea surface and $h \in \Re$ is the thermo-cline depth anomaly. Let Q be the Banach space of continuous $\Re \to \Re$ valued function defined on the interval J with the norm $\|(H, h)\| = \|H\| + \|h\|$ where $\|H\| = \sup\{|H(t) : t \in J|\}$ and $\|h\| = \sup\{|h(t) : t \in J|\}$. Particularly $Q = C(J) \times C(J)$, where $C(J)$ is the Banach space of continuous $\Re$ valued functions defined on the interval J with the sup norm.

Although, the above-stated model is not capable to classify the effect of memory and also the movement of ENSO at the different surface of the mechanism through the global motion is taking place. Hence, in order to get include these two above said effect into the mathematical formulation, we moderate the system by replacing the ordinary time derivative to the newly introduced Caputo–Fabrizio fractional derivative as given by

$$\begin{aligned} {}_0^{CF}D_t^{\alpha}H &= cH + \eta h - \varepsilon H^3, 0 < \alpha < 1 \\ {}_0^{CF}D_t^{\alpha}h &= -\theta H - \gamma h. \end{aligned} \tag{10}$$

And the initial conditions are the same as stated in Eq. (9).

4. Existence of the coupled–solutions of fractional ENSO model

In this portion, we present the existence of the coupled–solutions by using the fixed-point theorem. Applying the integral operator on Eq. (10), we get

$$\begin{aligned} H(t) - H(0) &= {}_0^{CF}I_t^{\alpha}\left\{cH(t) + \eta h(t) - \varepsilon H^3(t)\right\}, \\ h(t) - h(0) &= {}_0^{CF}I_t^{\alpha}\{-\theta H(t) - \gamma h(t)\}, \end{aligned} \tag{11}$$

Employing the notation suggested by Nieto and Losada [17], we procure

$$\begin{aligned} H(t) - H(0) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left\{cH(t) + \eta h(t) - \varepsilon H^3(t)\right\} \\ &\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left\{cH(y) + \eta h(y) - \varepsilon H^3(y)\right\}dy, \\ h(t) - h(0) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\{-\theta H(t) - \gamma h(t)\} \\ &\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \{-\theta H(y) - \gamma h(y)\}dy. \end{aligned} \tag{12}$$

For clarity, we interpret

$$\begin{aligned} K_1(t, H) &= cH(t) + \eta h(t) - \varepsilon H^3(t), \\ K_2(t, h) &= -\theta H(t) - \gamma h(t). \end{aligned} \tag{13}$$

Theorem 4.1. *K_1 and K_2 fulfill the Lipschiz condition and contraction if the succeeding inequality satisfies*

$$0 < \left(c + \varepsilon(a^2 + b^2 + ab)\right) \leq 1$$

Proof. We begin with K_1. Suppose H and G be two functions, then we assess the succeeding

$$\|K_1(t, H) - K_1(t, G)\| = \left\|c(H(t) - G(t)) - \varepsilon\left(H^3(t) - G^3(t)\right)\right\|. \tag{14}$$

Applying the triangular inequality of norm on Eq. (14), we get

$$\begin{aligned} \|K_1(t, H) - K_1(t, G)\| &\leq \|c(H(t) - G(t))\| + \varepsilon\left\|\left(H^3(t) - G^3(t)\right)\right\| \\ &\leq \|c(H(t) - G(t))\| + \varepsilon\|(H(t) - G(t)) \\ &\quad \left(H^2(t) + G^2(t) + H(t)G(t)\right)\| \\ &\leq \|c(H(t) - G(t))\| + \varepsilon\left\|(H(t) - G(t))\left(a^2 + b^2 + ab\right)\right\| \\ &\leq \left(c + \varepsilon\left(a^2 + b^2 + ab\right)\right)\|(H(t) - G(t))\|. \end{aligned} \tag{15}$$

Taking $A = (c + \varepsilon(a^2 + b^2 + ab))$, where $H(t)$ and $G(t)$ are bounded functions such that $\|H(t)\| \leq a$, $\|G(t)\| \leq b$, then we have

$$\|K_1(t, H) - K_1(t, G)\| \leq A\|H(t) - G(t)\|. \tag{16}$$

Hence, the Lipschiz condition is fulfilled for K_1 and if an additionally $0 < (c + \varepsilon(a^2 + b^2 + ab)) \leq 1$, then it is also a contraction.