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Feigenbaum's constants in reverse bifurcation of fractional-order Rössler system

Zengshan Li[a], Diyi Chen[a,b,*], Mengmeng Ma[a], Xinguang Zhang[b], Yonghong Wu[b]

[a] Department of Electrical Engineering, Northwest A&F University, Shaanxi Yangling 712100, P. R. China
[b] Department of Mathematics and Statistics, Faculty of Science and Engineering, Curtin University, Perth Western Australia 6845, Australia

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ABSTRACT

This paper demonstrates the existence of Feigenbaum's constants in reverse bifurcation for fractional-order Rössler system. First, the numerical algorithm of fractional-order Rössler system is presented. Then, the definition of Feigenbaum's constants in reverse bifurcation is provided. Third, in order to observe the effect of fractional-order to Feigenbaum's constants in reverse bifurcation, a series of bifurcation diagrams are computed. The Feigenbaum's constants in reverse bifurcation are measured and the error percentage in fractional-order Rössler system is presented. The simulation results show that Feigenbaum's constants exist in reverse bifurcation for fractional-order Rössler system. Especially, the Feigenbaum's constants still exist in the periodic windows. A summary on previous others' works about Feigenbaum's constants is proposed. This paper draw a conclusion that the constants are universal in both period-doubling bifurcation and reverse bifurcation for both integer and fractional-order system.

1. Introduction

Fractional calculus has appeared more than three hundred years ago, nevertheless, the evolution in studying fractional calculus takes place in recent years. Many interesting dynamical behaviors have been observed in a number of fractional-order nonlinear systems, such as fractional-order Chua's circuit [1–2], fractional-order Chen system [3–4], fractional-order jerk model [5], fractional-order cellular neural networks [6] and fractional-order Lorenz system [7–9]. The fractional-order nonlinear systems have attracted more and more researchers to study them [10–28].

The fact that Feigenbaum's constants exist in period-doubling bifurcation was discovered by Feigenbaum in 1975 [29]. It has been discovered that Feigenbaum's constants exist in many integer systems [30–31]. In 1980, Lorenz [32] put forward "reverse bifurcation". Compared with the period-doubling bifurcation, reverse bifurcation means 2^{i-1} chaotic bands split into 2^i chaotic bands (We introduce reverse bifurcation in detail in Section 3.1). Huberman and Rudnick [33] pointed out that Feigenbaum's constants are applied in reverse bifurcation as well, which was confirmed by Shau-Jin Chang and Jon Wright [34]. As for fractional-order chaotic systems, there are few contributions on Feigenbaum's constants. Chen et al [35] measured the Feigenbaum's constants in a continuous time fractional-order system; Ho et al [36] measured the Feigenbaum's constants in fractional degree Yin–Yang Henon map.

However, few published papers report Feigenbaum's constants in reverse bifurcation of fractional-order nonlinear system to our best knowledge. This paper studies Feigenbaum's constants in reverse bifurcation for fractional-order system in detail. The simulation results show that Feigenbaum's constants exist in reverse bifurcation for fractional-order Rössler system. This discovery is interesting in the framework of fractional-order chaotic systems. It may offer a new way to study the deep dynamics of fractional-order systems. Meanwhile, it is a bridge between integer-order system and fractional-order system.

Motivated by the above discussions, there shows two advantages which make our approach more attractive compared with the prior works. First, we study the Feigenbaum's constants in reverse bifurcation in fractional-order Rössler system, which is a pioneer work. More specially, we measure the Feigenbaum's constants in reverse bifurcation even in the periodic windows. Second, we do a conclusive summary on the scaling law discovered by Feigenbaum both in period-doubling bifurcation and in reverse bifurcation for both integer-order system and fractional-order system.

This paper is organized by following: Section 2 presents the numerical algorithm of the fractional-order Rössler system. In Section 3, we present the definition of Feigenbaum's constants in reverse bifurcation, measure the universal constants and do a conclusive summary on Feigenbaum's constants. The conclusion is drawn in Section 4.

* Corresponding author.
E-mail address: diyichen@nwsuaf.edu.cn (D. Chen).

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2. Numerical algorithm of Fractional-order Rössler system

The Rössler system is

$$\begin{cases} \dfrac{dx}{dt} = -y - z \\ \dfrac{dy}{dt} = x + ay \\ \dfrac{dz}{dt} = b + z(x - c) \end{cases}, \tag{1}$$

in which x, y, z are the state variables and a, b, c are the parameters. Thus, fractional-order Rössler system can be defined as

$$\begin{cases} D^{q_1}x = -y - z \\ D^{q_2}y = x + ay \\ D^{q_3}z = b + z(x - c) \end{cases}. \tag{2}$$

In above equation, D is the Caputo fractional derivative operator of order [37], which is defined as

$$D_*^q y(x) = J^{m-q} y^{(m)}(x), \quad q > 0 \tag{3}$$

where $m = \lceil q \rceil$, and $y^{(m)}$ is the ordinary mth derivative of y.

$$J^\theta z(x) = \frac{1}{\Gamma(\theta)} \int_0^x (x-t)^{\theta-1} z(t) dt \tag{4}$$

is the Riemann–Liouville integral operator with order $\theta > 0$, and $\Gamma(x)$ is the Euler's gamma function.

For the numerical algorithm of a fractional-order system, in this paper, the Adams-Bashforth-Moulton type predictor-corrector scheme is applied. This method, as a time domain approach, is more precise and effective, compared with the frequency domain approach. The Adams-Bashforth-Moulton type predictor-corrector scheme is based on the following fractional differential equation

$$D_t^q y(t) = f(y(t), t), y^{(k)}(0) = y_0^k, k = 0, 1, \ldots, m-1 \tag{5}$$

which is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{\lceil q \rceil - 1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau, \ y(\tau)) d\tau. \tag{6}$$

Discretizing the Volterra equation by setting $t_n = nh$ ($n = 0, 1, \ldots, N$) and $h = T_{sim}/N$, we can obtain

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^q}{\Gamma(q+2)} f(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_n(t_j)), \tag{7}$$

where

$$a_{j,b+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0, \\ (n-j+2)^{q+1} + (n-j)^{q+1} \\ \quad -2(n-j+1)^{q+1}, & 1 \le j \le n, \\ 1, & j = n+1. \end{cases} \tag{8}$$

The predictor $y_h^p(t_{n+1})$ is given by

$$y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^k + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_n(t_j)), \tag{9}$$

in which

$$b_{j,n+1} = \frac{h^q}{q}((n+1-j)^q - (n-j)^q). \tag{10}$$

The error estimate is $\max_{i=0,1,\ldots,N} |y(t_i) - y_h(t_i)| = O(h^p)$, in which $p = \min(2, 1+q)$.

Employing the Adams-Bashforth-Moulton scheme, the fractional-order Rössler system can be presented as follows:

$$\begin{cases} x_{n+1} = x_0 + \dfrac{h^{q_1}}{\Gamma(q_1+2)} \left\{ [-y_{n+1}^p - z_{n+1}^p] + \sum_{j=0}^{n} a_{1,j,n+1}(-y_j - z_j) \right\} \\ y_{n+1} = y_0 + \dfrac{h^{q_2}}{\Gamma(q_2+2)} \left\{ [x_{n+1}^p + a y_{n+1}^p] + \sum_{j=0}^{n} a_{2,j,n+1}(x_j + a y_j) \right\} \\ z_{n+1} = z_0 + \dfrac{h^{q_3}}{\Gamma(q_3+2)} \left\{ b + z_{n+1}^p (x_{n+1}^p - c) + \sum_{j=0}^{n} a_{3,j,n+1}(b + z_j(x_j - c)) \right\} \end{cases} \tag{11}$$

in which

$$\begin{cases} x_{n+1}^p = x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^{n} b_{1,j,n+1}(-y_j - z_j) \\ y_{n+1}^p = y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^{n} b_{2,j,n+1}(x_j + a y_j) \\ z_{n+1}^p = z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=0}^{n} b_{3,j,n+1}(b + z_j(x_j - c)) \end{cases}, \tag{12}$$

$$\begin{cases} b_{1,j,n+1} = \dfrac{h^{q_1}}{q_1}((n-j+1)^{q_1} - (n-j)^{q_1}), & 0 \le j \le n \\ b_{2,j,n+1} = \dfrac{h^{q_2}}{q_2}((n-j+1)^{q_2} - (n-j)^{q_2}), & 0 \le j \le n \\ b_{3,j,n+1} = \dfrac{h^{q_3}}{q_3}((n-j+1)^{q_3} - (n-j)^{q_3}), & 0 \le j \le n \end{cases} \tag{13}$$

and

$$\begin{cases} a_{1,j,n+1} = \begin{cases} n^{q_1} - (n-q_1)(n+1)^{q_1} & j = 0 \\ (n-j+2)^{q_1+1} + (n-j)^{q_1+1} \\ \quad -2(n-j+1)^{q_1+1} & 0 \le j \le n \end{cases} \\ a_{2,j,n+1} = \begin{cases} n^{q_2} - (n-q_2)(n+1)^{q_2} & j = 0 \\ (n-j+2)^{q_2+1} + (n-j)^{q_2+1} \\ \quad -2(n-j+1)^{q_2+1} & 0 \le j \le n \end{cases} \\ a_{3,j,n+1} = \begin{cases} n^{q_3} - (n-q_3)(n+1)^{q_3} & j = 0 \\ (n-j+2)^{q_3+1} + (n-j)^{q_3+1} \\ \quad -2(n-j+1)^{q_3+1} & 0 \le j \le n \end{cases} \end{cases}. \tag{14}$$

3. Feigenbaum's constants in reverse bifurcation of fractional-order Rössler system

In this section, we will focus on the Feigenbaum's constants in reverse bifurcation of fractional-order Rössler system. At first, the definition of Feigenbaum's constants in reverse bifurcation will be elaborated explicitly. Next, we will obtain the Feigenbaum's constants in reverse bifurcation and the error percentage by analyzing a series of bifurcation diagrams of integer and fractional-order Rössler system. Finally, we do a summary about Feigenbaum's constants.

3.1. Definition of Feigenbaum's constants in reverse bifurcation

(i) Feigenbaum's constants in period-doubling bifurcation.

The first and the second Feigenbaum's constants are defined as

$$\delta = \lim_{n \to \infty} \frac{a_{n+1} - a_n}{a_{n+2} - a_{n+1}} \tag{15}$$

and

$$|\alpha| = \left| \frac{\Delta M_{x_i}}{\Delta M_{x_{i+1}}} \right|, \tag{16}$$

respectively, where a_n is the value of the parameter at the nth period-doubling bifurcation point; $|\Delta M_{x_i}|$ is the width of the widest bifurcation fork of the ith period-doubling bifurcation [29]. More specifically, $\delta = 4.6692016\ldots$, $|\alpha| = 2.5029078\ldots$.