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Lyapunov exponents and poles in a non Hermitian dynamics

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ABSTRACT

By means of expressing volumes in phase space in terms of traces of quantum operators, a relationship between the poles of the scattering matrix and the Lyapunov exponents in a non Hermitian quantum dynamics, is presented. We illustrate the formalism by characterizing the behavior of the Gamow model whose dissipative decay time, measured by its decoherence time, is found to be inversely proportional to the Lyapunov exponents of the unstable periodic orbits. The results are in agreement with those obtained by means of the semiclassical periodic–orbit approach in quantum resonances theory but using a simpler mathematics.

1. Introduction

The interest in the study of non Hermitian Hamiltonians is related with the interpretation of phenomena such as nuclear resonances, dissipation, relaxation of nonequilibrium states, typical of open systems. In scattering systems one can consider quantum resonances, called “quasi-stationary states” or “Gamow states”, instead of scattering solutions [1–5]. Gamow states play in open systems a similar role as the eigenstates of closed systems and their eigenvalues are complex numbers with non zero imaginary part. Moreover, they characterize the unstable periodic orbits and are physically interpreted as particle–states transferred from the system to its environment. Any measurement on a open system drastically changes its properties by converting discrete energy levels into decaying Gamow states, which can be described by a non Hermitian Hamiltonian [6–9]. In this context, the characteristic decay times are given by the imaginary part of the complex eigenvalues, i.e. the so called *poles of the scattering matrix* [3]. These arise as a result of the analytic extension of a Hamiltonian whose degeneration makes the perturbation theory inapplicable [10–19]. Furthermore, non Hermitian Hamiltonians allow to describe the non-unitary time evolutions that appear in open quantum systems [9]. Properties of open quantum systems like nonequilibrium phenomena and dissipation can be characterized by the positivity of the Kolmogorov–Sinai entropy which, in turn, is equal to the sum of all positive Lyapunov exponents due to the Pesin theorem [20–23]. The characteristic time of these kind of processes is given by the Kolmogorov–Sinai time, which provides a decay time in the phase space as a function of the Lyapunov exponents [24–26]. In addition, in chaotic open quantum systems the Lyapunov exponents and the escape rates of classical trajectories have been characterized by means of semiclassical techniques [27–30], and also from the strategy of ranking chaos looking at the decay of correlations between states and observables [31,32].

The present contribution shows a novel way of obtaining Lyapunov exponents in terms of poles of the scattering matrix (S–matrix) in non Hermitian Hamiltonian systems, but with a simpler mathematics than the used in the literature. As a consequence of this study, the following is obtained: i) a method for obtaining the part of the KS–entropy free of the escape rates in open quantum systems [29], and ii) conditionally invariant measures describing classical localization of chaotic states [30]. The dynamical indicator we choose to obtain our results is the Kolmogorov–Sinai entropy by two reasons, mainly. The first is that due to the Pesin theorem and the relationship between the KS–entropy and the KS–time, the sum of the Lyapunov exponents can be expressed in terms of the KS–time which is the time that a little volume takes to spread throughout all the phase space [24–26]. In turn, with the help of the Wigner transformation the evolution of volumes in phase space can be written as quantum mean values, that decay according to the lifetimes given by the poles of the S–matrix. Thus, KS–entropy serves an intermediate tool to connect Lyapunov exponents with poles. Secondly, the robustness of the KS-entropy guarantees the validity of the results for a wide range in the initial conditions, as we shall see.

Using the idea of expressing classical quantities in terms of traces of quantum operators as in Gomez and Castagnino [31],

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Gomez et al. [32], Castagnino and Lombardi [33], Gomez and Castagnino [34,35], we present a relationship between the poles of the scattering matrix and the Lyapunov exponents in a non Hermitian quantum dynamics, where the Kolmogorov–Sinai time expresses the contractions and expansions of volumes in the phase space along their dynamics. The paper is organized as follows. In Section 2 we give the preliminaries and the mathematical formalism. In Section 3 we express the Lyapunov exponents in terms of the poles by means of the non–unitary evolution of a little volume element in phase space. In Section 4 we illustrate the formalism by applying it to the Gamow model. In Section 5 we discuss the results with regard the quantum resonances theory. Finally, in Section 6 some conclusions and future research directions are outlined.

2. Preliminaries

2.1. Kolmogorov–Sinai time and Pesin theorem

The characteristic time for a nonequilibrium process in a mixing dynamics is the Kolmogorov–Sinai time (KS–time) τ_{KS}, which measures the necessary time to take a number of initially close phase points to uniformly distribute over the energy surface. Moreover, τ_{KS} is inversely proportional to the Kolmogorov–Sinai entropy (KS–entropy), denoted by h_{KS}

$$\tau_{KS} = \frac{1}{h_{KS}} \tag{1}$$

Another important property is the relationship between the maximum Lyapunov exponent and h_{KS}. Krylov observed that a little phase volume ΔV after a time t will be spread over a region with a volume $\Delta V(t) = \Delta V \exp(h_{KS} t)$ where $\Delta V(t)$ is of order 1 [24,25]. This means that after a time

$$t_0 = \frac{1}{h_{KS}} \ln \frac{1}{\Delta V} \tag{2}$$

the initial phase volume ΔV is spread over the whole phase space. Consequently, one might expect that the typical relaxation times are proportional to $\frac{1}{h_{KS}}$.

On the other hand, the Pesin theorem relates the KS-entropy h_{KS} with the Lyapunov exponents by means of the formula [20–23]

$$h_{KS} = \int_{\Gamma} \sum_{\sigma_i > 0} \sigma_i(q, p) dq dp \tag{3}$$

where Γ is the phase space. For the special case when the σ_i are constant over all phase space one has

$$h_{KS} = \sum_{\sigma_i > 0} \sigma_i \tag{4}$$

It should be noted the interest of the formula (3) and its physical meaning. Pesin theorem relates the KS-entropy, that is the average unpredictability of information of all possible trajectories in the phase space, with the exponential instability of motion. Then, the main content of Pesin theorem is that $h_{KS} > 0$ is a sufficient condition for the chaotic motion. Using Eqs. (1) and (4) one obtains the following relationship between τ_{KS} and the Lyapunov exponents

$$\frac{1}{\tau_{KS}} = \sum_{\sigma_i > 0} \sigma_i \tag{5}$$

In the following sections we will use this formula in order to obtain a relationship between the Lyapunov exponents and the poles of the S–matrix, within the context of effective non Hermitian Hamiltonians.

2.2. Wigner transformation

We recall some properties of the Wigner transformation formalism [36–39] we will use throughout the paper. Given a quantum operator $\hat{A}$ the Wigner transformation $W_{\hat{A}} : \mathbb{R}^{2M} \mapsto \mathbb{R}$ of $\hat{A}$ is defined by

$$W_{\hat{A}}(q, p) = \frac{1}{h^M} \int_{\mathbb{R}^M} \langle q + \Delta | \hat{A} | q - \Delta \rangle e^{2i \frac{p\Delta}{\hbar}} d\Delta \tag{6}$$

where $q, p, \Delta \in \mathbb{R}^M$. The Weyl symbol $\widetilde{W}_{\hat{A}} : \mathbb{R}^{2M} \mapsto \mathbb{R}$ of $\hat{A}$ is defined by $\widetilde{W}_{\hat{A}}(q, p) = \hbar^M W_{\hat{B}}(q, p)$ where $\hbar = \frac{h}{2\pi}$ and h is the Planck constant. In particular, for the identity operator $\hat{I}$ one has $\widetilde{W}_{\hat{I}}(q, p) = 1(q, p)$ where $1(q, p)$ is the function that is constantly equal to 1. One of the main properties of the Wigner transformation is the expression of integrals over the phase space in terms of trace of operators by means of [37]

$$\mathrm{Tr}(\hat{A}\hat{B}) = \int_{\mathbb{R}^{2M}} W_{\hat{A}}(q, p) \widetilde{W}_{\hat{B}}(q, p) dq dp \tag{7}$$

valid for all pair of operators $\hat{A}, \hat{B}$ where $\hat{A}\hat{B}$ denotes the product of $\hat{A}$ and $\hat{B}$ and $\mathrm{Tr}(\ldots)$ is the trace operation. Using the definition of the Weyl symbol it can be shown the following result that relates the Weyl symbols of an operator and of the same but evolved at a time t. The proof can be found in the Appendix.

Lemma 2.1. *Let $\widetilde{W}_{\hat{A}}(q, p)$ be the Weyl symbol of an operator $\hat{A}$. Then the Weyl symbol of $\hat{A}(-t) = \hat{U}_t^\dagger \hat{A} \hat{U}_t$ is $\widetilde{W}_{\hat{A}}(q(t), p(t))$ where $(q(t), p(t)) = (T_t q, T_t p)$ and T_t is the classical evolution given by Hamilton equations. For all $t \in \mathbb{R}$ one has*

$$\widetilde{W}_{\hat{U}_t^\dagger \hat{A} \hat{U}_t}(q, p) = \widetilde{W}_{\hat{A}}(q(t), p(t)) \quad \forall\ (q, p) \in \mathbb{R}^2 \tag{8}$$

where $\hat{A}(-t) = \hat{U}_{-t} \hat{A} \hat{U}_{-t}^\dagger$, $\hat{U}_t = e^{-i \frac{\hat{H}}{\hbar} t}$ is the evolution operator, and $\hat{U}_t^\dagger$ is the Hermitian conjugate of $\hat{U}_t$.

2.3. Scattering matrix and analytic continuations

The motivations for the use of non Hermitian Hamiltonians arise naturally when modeling phenomena of nuclear physics or decay processes by means of scattering theory [3,9]. Mathematically, these are obtained by the *analytic dilation method* [40]. For instance, in the context of microwave billiards it is well known that the spectrum is modified by the presence of the coupling antennas, where the quantum probability amplitude that a certain entering state $|\psi_{in}\rangle$ is scattered into an outgoing state $|\psi_{out}\rangle$ is given by the scattering matrix $\hat{S}$

$$|\psi\rangle = |\psi_{in}\rangle + \hat{S}|\psi_{out}\rangle \tag{9}$$

If $\hat{H} = \hat{H}_0 + \hat{V}$ is the total Hamiltonian of the system with $\hat{H}_0$ the undisturbed Hamiltonian and $\hat{V}$ the potential of interaction, then it can be shown that $\hat{S}$ takes the form [3]

$$\hat{S} = \hat{1} - 2i\hat{W}^\dagger \frac{1}{E - \hat{H}_0 + \hat{W}\hat{W}^\dagger} \hat{W} \tag{10}$$

where $\hat{W}$ contains the information on the coupling strengths between the unperturbed states and the resonances, and it can be given in terms of the potential $\hat{V}$. Thus, the poles of $\hat{S}$ are the eigenvalues of the effective non Hermitian Hamiltonian

$$\hat{H}_0 - i\hat{W}\hat{W}^\dagger \tag{11}$$

This type of effective Hamiltonian have been widely used in nuclear physics [1,2]. In particular, if E_n^0 is the nth eigenvalue of $\hat{H}_0$ then in the limiting case of small coupling strengths the eigenvalues are given in first order perturbation theory by

$$E_n = E_n^0 - i\left(\hat{W}\hat{W}^\dagger\right)_{nn} = E_n^0 - i\sum_k |W_{nk}|^2 \tag{12}$$