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Existence of positive solution for a cross-diffusion predator-prey system with Holling type-II functional response☆

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ABSTRACT

This paper is purported to investigate a cross-diffusion system arising in a predator-prey population model including Holling type-II functional response in a bounded domain with Dirichlet boundary condition. The asymptotical stabilities are investigated to this system by using the method of eigenvalue. Moreover, the existence of positive steady states are considered by using fixed points index theory, bifurcation theory, energy estimates and the differential method of implicit function.

1. Introduction

For traditional model of predator-prey, the authors do not usually take into account either the fact that the distribution of population is usually inhomogeneous or the fact that the species develop naturally strategies for survival. Both of these considerations involve diffusion process which is complicated when the concentration levels of species cause different movements of population. Such movements arise from the concentration of other species or that of the same species. The natural tendency of species to areas of smaller population concentration produces random diffusion, while the movement of species in response to behavior of another population, for example, pursuit and evasion, gives rise to the complicated diffusion which is modeled by cross-diffusion. This more intricate but realistic predator-prey model has been proposed by ecologists and mathematicians [14,16,17].

In this article, we investigate the following elliptic system with cross-diffusion incorporating Holling type-II functional response under the Dirichlet boundary condition:

$$\begin{aligned} -\Delta[(1+\alpha v)u] &= u\left(a-u-\frac{cv}{1+mu}\right) \quad \text{in } \Omega,\\ -\Delta\left[\left(\mu+\frac{1}{1+\beta u}\right)v\right] &= v\left(b-v+\frac{ecu}{1+mu}\right) \quad \text{in } \Omega,\\ u=v&=0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $R^N(N\geq 1$ is an integer) with a smooth boundary $\partial\Omega$; u and v represent the densities of the prey and predator respectively. The parameters α, β, μ, a, b, c, e and m are positive constants. The model (1.1) means that, in addition to the dispersive force, the diffusion also depends on population pressure from other species.

The flux of diffusion to the predators of the system (1.1) is

$$\begin{aligned} J &= -\nabla\left(\mu+\frac{1}{1+\beta u}\right)v\\ &= -\left(\mu+\frac{1}{1+\beta u}\right)\nabla v+\frac{\beta v}{(1+\beta u)^2}\nabla u. \end{aligned}$$

The part $\frac{\beta v}{(1+\beta u)^2}\nabla u$ of the diffusion flux is directed toward the increasing densities of the prey, which indicates that the preys respond to attack of team for the movement of predators. The part $-(\mu+\frac{1}{1+\beta u})\nabla v$ of the diffusion flux is directed toward the decreasing densities of the predators, which implies that the predators move towards the preys to predate [1].

When the predators invade areas with high food abundance to increase the efficiency of foraging and the preys switch to defend or run away, these movements of diffusion demonstrate rich dynamics. Stability and existence of positive solution are hot topics to discuss for the predator-prey system. Many authors have established the existence of positive solution and stability in various population dynamics models [7,8,10,15,18,21–23], but most of them are without cross-diffusion. Especially, to my knowledge, there are few papers to investigate the stability and existence of the positive solution for the cross-diffusion predator-prey system under Dirichlet boundary condition. In this paper, we shall demonstrate asymptotical stability of system (1.1) by using the method of eigenvalue, and the existence of positive steady states for (1.1) by employing

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fixed points index theory, bifurcation theory, energy estimates and the differential method of implicit function.

This paper is organized into five sections. In next section, the preliminaries are presented. In Section 3, the asymptotical stabilities are established for this cross-diffusion model. In section 4, the sufficient conditions to existence of positive solutions of system (1.1) are found. In Section 5, global bifurcation of positive solutions is investigated to system (1.1). In the final section, we make a brief comments and conclusions.

2. Preliminaries

Denote by $\lambda_1(q) < \lambda_2(q) \le \lambda_3(q) \le \cdots$ all eigenvalues of

$$\begin{aligned} -\Delta u + q(x)u &= \lambda u && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $q(x) \in C(\overline{\Omega})$. For convenience, we denote $\lambda_i = \lambda_i(0)$. It is well known that $\lambda_i(q_1) < \lambda_i(q_2)$ when $q_1(x) \le q_2(x)$ and $q_1(x)$ is not equivalent to $q_2(x)$, and $\lambda_1(q)$ is simple. Moreover, by the Proposition A.1 in [9], the principal eigenvalue $\lambda_1(q)$ has the following properties:

Proposition 2.1.

(i) *Assume that $\phi_n \in H_0^1(\Omega)$ and $\phi \in H_0^1(\Omega)$ are the corresponding eigenfunctions of (2.1) and satisfy $\|\phi_n\|_{L^2(\Omega)} = \|\phi\|_{L^2(\Omega)} = 1$ for $q_n \in C(\overline{\Omega})$ and $q \in C(\overline{\Omega})$. If $\lim_{n\to\infty} \|q_n - q\|_{L^\infty(\Omega)} = 0$, then $\lim_{n\to\infty} \lambda_1(q_n) = \lambda_1(q)$ and $\lim_{n\to\infty} \phi_n = \phi$ strongly in $H_0^1(\Omega)$;*

(ii) *Let a mapping $\xi \to q_\xi$ be differentiable continuously from an open interval (c, d) to $C(\overline{\Omega})$ with respect to supremum norm. If $\phi_\xi \in H_0^1(\Omega)$ is the unique positive eigenfunction corresponding to $\lambda_1(q_\xi)$ with $\|\phi_\xi\|_{L^2(\Omega)} = 1$, then $\xi \to \lambda_1(q_\xi)$ is differentiable continuously from (c, d) to R and*

$$\frac{d}{d\xi}\lambda_1(q_\xi) = \int_\Omega \frac{\partial q_\xi}{\partial \xi}\phi_\xi^2 dx.$$

Consider the following logistic equation with random diffusion

$$\begin{aligned} -\Delta u &= u(\rho - u) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where ρ is a positive constant and $\Omega (\Omega \subset R^n)$ is a bounded open set with smooth boundary $\partial\Omega$. Then, by Lemma 1 in [4] and Propositions 6.1–6.4 in [6], we obtain the following proposition:

Proposition 2.2.

(i) *There does not exist nontrivial solution when $\rho \le \lambda_1$;*

(ii) *When $\rho > \lambda_1$, then (2.2) admits a unique positive solution $\theta_\rho(x)$ satisfying $0 < \theta_\rho(x) < \rho$ for all $x \in \Omega$;*

(iii) *The unique positive solution θ_ρ of (2.2) satisfies: $\lim_{\rho\to\lambda_1^+} \theta_\rho = 0$ uniformly in Ω, $\lim_{\rho\to\infty} \theta_\rho = \infty$ and $\lim_{\rho\to\infty} \theta_\rho/\rho = 1$ uniformly in K, where K is any compact subset of Ω;*

(iv) *The mapping $\rho \to \theta_\rho$ is C^1 from (λ_1, ∞) to $C(\overline{\Omega})$ and $\theta_\rho(x)$ increases strictly with respect to ρ. More precisely,*

$$\frac{\partial \theta_\rho}{\partial \rho} = (-\Delta + (2\theta_\rho - \rho)I)^{-1}\theta_\rho,$$

where $(-\Delta + (2\theta_\rho - \rho)I)^{-1}$ is the inverse operator of $-\Delta + (2\theta_\rho - \rho)I$ with zero Dirichlet boundary condition.

Let W be the natural positive cone of E, where E is a Banach space. Define $W_y = \{x \in E : y + \kappa x \in W$ for some $\kappa > 0\}$ and $S_y = \{x \in \overline{W}_y : -x \in \overline{W}_y\}$ for $y \in W$. Suppose that y_* is a fixed point of compact operator A: $W \to W$ and $L = A'(y_*)$ is the Fréchet derivative of A at y_*. If there exist $t \in (0, 1)$ and $\omega \in \overline{W}_{y_*} \setminus S_{y_*}$ such that $\omega - tA'\omega \in S_{y_*}$, then we call that A' has property γ on $\overline{W}_{y_*}$. Define $\text{index}_W(A, U) = \text{index}(A, U, W) = \deg_W(I - A, U, 0)$ for an open subset $U \subset W$, where I is the identity mapping. Moreover, the fixed point index of A at y_* is defined by $\text{index}_W(A, y_*) = \text{index}(A, y_*, W) = \text{index}(A, U_{y_*}, W)$ in W, where $U(y_*)$ is a small open neighborhood of y_* in W. Then the following lemma can be derived from the results in [5,8,12,13].

Proposition 2.3. *Let $I - L$ be invertible on $\overline{W}_{y_*}$.*

(i) *If L has property γ on $\overline{W}_{y_*}$, then $\text{index}_W(A, y_*) = 0$;*

(ii) *If L does not have property γ on $\overline{W}_{y_*}$, then $\text{index}_W(A, y_*) = (-1)^\delta$, where δ is the sum of algebraic multiplicities of the eigenvalues of L which are greater than 1.*

For the system (2.1), by Lemmas 2.1 and 2.3 in [12], we have the following results.

Proposition 2.4. *Assume that M is a positive constant such that $M - q(x) > 0$ on $\overline{\Omega}$ for $q \in C^\alpha(\overline{\Omega})$. Then*

(i) $\lambda_1(q) > 0 \Rightarrow r[(M - \Delta)^{-1}(M - q(x))] < 1$;

(ii) $\lambda_1(q) < 0 \Rightarrow r[(M - \Delta)^{-1}(M - q(x))] > 1$;

(iii) $\lambda_1(q) = 0 \Rightarrow r[(M - \Delta)^{-1}(M - q(x))] = 1$,
where $r[(M - \Delta)^{-1}(M - q(x))]$ is the spectral radius of the linear operator $(M - \Delta)^{-1}(M - q(x))$.

3. Analysis of the trivial and semi-trivial solution of (1.1)

It is easy to check that the system (1.1) admits the trivial solution (0, 0), and two semi-trivial solutions $(\theta_a, 0)$ if $a > \lambda_1$ and $(0, (\mu+1)\theta_{b/(\mu+1)})$ if $b > (\mu+1)\lambda_1$, where $\theta_{b/(\mu+1)}$ is a positive of (2.2) with ρ replaced by $b/(\mu+1)$. The following theorem is the main result in this section.

Theorem 3.1.

(i) *Assume that $a < \lambda_1$ and $b < (\mu+1)\lambda_1$, then trivial steady state (0, 0) is locally asymptotically stable, while (0, 0) is unstable if $a > \lambda_1$ or $b > (\mu+1)\lambda_1$;*

(ii) *There exists μ^* such that semi-trivial steady state $(\theta_a, 0)$ is locally asymptotically stable when $\mu > \mu^*$;*

(iii) *Assume that $\lambda_1(\frac{c(\mu+1)\theta_{b/(\mu+1)} - a}{1 + \alpha(\mu+1)\theta_{b/(\mu+1)}}) > 0$, then semi-trivial $(0, (\mu+1)\theta_{b/(\mu+1)})$ (which exists when $b > (\mu+1)\lambda_1$) is locally asymptotically stable, while it is unstable if $\lambda_1(\frac{c(\mu+1)\theta_{b/(\mu+1)} - a}{1 + \alpha(\mu+1)\theta_{b/(\mu+1)}}) < 0$.*

Proof.

(i) The proof of (i) is similar to that of (ii), so we omit it.

(ii) The linearized model of (1.1) at $(\theta_a, 0)$ is as follows:

$$\begin{aligned} -\Delta(u + \alpha\theta_a v) &= (a - 2\theta_a)u - \tfrac{c\theta_a}{1+m\theta_a}v && \text{in } \Omega,\\ -\Delta\left(\mu + \tfrac{1}{1+\beta\theta_a}\right)v &= \left(b + \tfrac{ec\theta_a}{1+m\theta_a}\right)v && \text{in } \Omega,\\ u = v &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

The stability of $(\theta_a, 0)$ is determined by the following eigenvalue problem

$$\begin{aligned} -\Delta(\phi + \alpha\theta_a\psi) + (2\theta_a - a)\phi + \tfrac{c\theta_a}{1+m\theta_a}\psi &= \lambda\phi && \text{in } \Omega,\\ -\Delta\left(\mu + \tfrac{1}{1+\beta\theta_a}\right)\psi - \left(b + \tfrac{ec\theta_a}{1+m\theta_a}\right)\psi &= \lambda\psi && \text{in } \Omega,\\ \phi = \psi &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

The system (3.2) is not completely coupled, so it can be divided into the following the eigenvalue problems (see [11])

$$\begin{aligned} -\Delta\phi + (2\theta_a - a)\phi &= \lambda\phi && \text{in } \Omega,\\ \phi &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.3}$$