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Application of rotational spectrum for correlation dimension estimation

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ABSTRACT

Correlation dimension is one of the many types of fractal dimension. It is usually estimated from a finite number of points from a fractal set using correlation sum and regression in a log-log plot. However, this traditional approach requires a large amount of data and often leads to a biased estimate. The novel approach proposed here can be used for the estimation of the correlation dimension in a frequency domain using the power spectrum of the investigated fractal set. This work presents a new spectral characteristic called “rotational spectrum” and shows its properties in relation to the correlation dimension. The theoretical results can be directly applied to uniformly distributed samples from a given point set. The efficiency of the proposed method was tested on sets with a known correlation dimension using Monte Carlo simulation. The simulation results showed that this method can provide an unbiased estimation for many types of fractal sets.

1. Introduction

Correlation dimension D_2 is a popular tool for fractal dimension estimation and belongs to a family of entropy-based fractal dimensions such as capacity dimension D_0, information dimension D_1 and their generalisation, Renyi dimension D_α, for $\alpha \geq 0$. The properties of the different dimension types are summarised in [1] and [2]. The main idea of using correlation dimension is the distance between its points in space. In the original concept, only the number of points that are not farther apart as a fixed value can carry the information about the density of points contained in the investigated set. The geometrical meaning of correlation dimension is explained well in [3].

This traditional approach of correlation dimension estimation is based on Grassberger and Procaccia's algorithm [4,5] and is widely used in biomedicine for electroencephalography signal analysis [6,7] or in cardiology [8]. Recently, new approaches of correlation dimension estimation were presented using a weighting function [9] or methods suitable for high-dimensional signals [10]. The linear regression model, on which the majority of methods are based, provides an often biased estimate of fractal dimension; for this reason, Hongying and Duanfeng [11] made some efforts to improve this procedure.

In this work, we present a novel approach of correlation dimension estimation that is based on the rotation of the power spectrum of a point set. The proposed method is stable even for a small number of points, and the resulting characteristic has a smooth development.

2. Correlation dimension

Correlation dimension, introduced by Grassberger and Procaccia, involves measuring the distance between all pairs of points in the investigated set. For the Lebesgue measurable set $\mathcal{F} \subset \mathbb{R}^n$, the *correlation sum* [4] is defined for $r > 0$ as the limit case

$$\mathrm{C}(r) = \lim_{N\to\infty} \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mathrm{I}(\|\boldsymbol{x_i} - \boldsymbol{x_j}\| \leq r), \tag{1}$$

where $\|\cdot\|$ denotes a Euclidean norm that is rotation invariant, I is the indicator function and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ are vectors from $\mathcal{F}$. Because the correlation dimension expresses the relative amount of points whose distance is less than r, the correlation sum can be rewritten as

$$\mathrm{C}(r) = \mathop{\mathrm{E}}_{\boldsymbol{x},\boldsymbol{y}\sim\mathrm{U}(\mathcal{F})} \mathrm{I}(\|\mathbf{x} - \mathbf{y}\| \leq r) = \mathop{\mathrm{prob}}_{\boldsymbol{x},\boldsymbol{y}\sim\mathrm{U}(\mathcal{F})} (\|\boldsymbol{x} - \boldsymbol{y}\| \leq r), \tag{2}$$

for $\boldsymbol{x}, \boldsymbol{y}$ that are uniformly distributed on $\mathcal{F}$. Therefore, C(r) is a cumulative distribution function of random variable $r = \|\boldsymbol{x} - \boldsymbol{y}\|$. The

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correlation dimension D_2 of set $\mathcal{F}$ is based on the correlation sum and is defined as

$$D_2 = \lim_{r\to 0^+} \frac{\ln \mathrm{C}(r)}{\ln r}, \tag{3}$$

if the limit exists.

3. Continuous spectrum of a point set

The Fourier transform of an n-dimensional set $\mathcal{F} \subset \mathbb{R}^n$ is defined using the operator of the expected value [12] as

$$\mathrm{F}(\boldsymbol{\omega}) = \mathop{\mathrm{E}}_{\boldsymbol{x}\sim \mathrm{U}(\mathcal{F})} \exp(-\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{x}) \tag{4}$$

for angular frequency $\boldsymbol{\omega} \in \mathbb{R}^n$ and for $\boldsymbol{x}$ uniformly distributed on $\mathcal{F}$. The *power spectrum* of set $\mathcal{F}$ equals $\mathrm{P}(\boldsymbol{\omega}) = |\mathrm{F}(\boldsymbol{\omega})|^2 = \mathrm{F}(\boldsymbol{\omega})\cdot\mathrm{F}^*(\boldsymbol{\omega})$, where F^* is a complex conjugate of F. Moreover, it can be expressed as

$$\begin{aligned}\mathrm{P}(\boldsymbol{\omega}) &= \mathop{\mathrm{E}}_{\boldsymbol{x}\sim \mathrm{U}(\mathcal{F})}\ \mathop{\mathrm{E}}_{\boldsymbol{y}\sim \mathrm{U}(\mathcal{F})} \exp(-\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{x})\exp(\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{y})\\ &= \mathop{\mathrm{E}}_{\boldsymbol{x},\boldsymbol{y}\sim \mathrm{U}(\mathcal{F})} \exp(-\mathrm{i}\boldsymbol{\omega}\cdot(\boldsymbol{x}-\boldsymbol{y})),\end{aligned} \tag{5}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are independent and identically distributed from $\mathcal{F}$. The power spectrum is frequently used for fractal set investigation [13–15]. When the research is physically motivated, it is usual to denote the angular frequency as $\omega = 2\pi/\lambda$ for wavelength λ of an X-ray or light beam.

4. Rotational spectrum

The goal of the novel method is to obtain a one-dimensional function as a derivative of the power spectrum, which is useful in fractal analysis. The procedure was inspired by Debye [16] and by his X-ray diffraction method, which is often referred to as the Debye-Scherrer method. We denote $\mathrm{SO}(n)$ as the group of all rotations in $\mathbb{R}^n$ around the origin. Because any rotation $\mathrm{R} \in \mathrm{SO}(n)$ is a linear transform, the following equation holds

$$\mathrm{R}(\boldsymbol{x}) - \mathrm{R}(\boldsymbol{y}) = \mathrm{R}(\boldsymbol{x}-\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|\cdot\boldsymbol{\xi}, \tag{6}$$

where $\boldsymbol{\xi}$ is a direction vector satisfying $\|\boldsymbol{\xi}\| = 1$ and $\boldsymbol{\xi} \in \mathcal{S}_{n-1}$ for an n-dimensional sphere $\mathcal{S}_{n-1} = \{\boldsymbol{x}\in\mathbb{R}^n : \|\boldsymbol{x}\| = 1\}$. Using the factorisation of angular frequency $\boldsymbol{\omega} = \Omega\cdot\boldsymbol{\psi}$ for $\Omega \in \mathbb{R}_0^+$ and normalisation vector $\boldsymbol{\psi} \in \mathcal{S}_{n-1}$, we can define *rotational spectrum* as

$$\mathrm{S}(\Omega) = \mathop{\mathrm{E}}_{\mathrm{R}\in\mathrm{SO}(n)}\ \mathop{\mathrm{E}}_{\boldsymbol{\psi}\in\mathcal{S}_{n-1}}\ \mathop{\mathrm{E}}_{\boldsymbol{x},\boldsymbol{y}\sim\mathrm{U}(\mathcal{F})} \exp(-\mathrm{i}\Omega\boldsymbol{\psi}\mathrm{R}(\boldsymbol{x}-\boldsymbol{y})), \tag{7}$$

which can be expressed explicitly in the following theorem.

Theorem 1. *Rotational spectrum can be expressed as*

$$\mathrm{S}(\Omega) = \mathop{\mathrm{E}}_{\boldsymbol{x},\boldsymbol{y}\sim\mathrm{U}(\mathcal{F})} \mathrm{H}_n(\Omega\|\boldsymbol{x}-\boldsymbol{y}\|), \tag{8}$$

where

$$\mathrm{H}_n(q) = \frac{2^{\frac{n-2}{2}}\cdot\Gamma\left(\frac{n}{2}\right)}{q^{\frac{n-2}{2}}}\mathrm{J}_{\frac{n-2}{2}}(q). \tag{9}$$

Proof. Because every rotation is a linear transform, we can rewrite the rotational spectrum as

$$\mathrm{S}(\Omega) = \mathop{\mathrm{E}}_{\boldsymbol{x},\boldsymbol{y}\sim\mathrm{U}(\mathcal{F})}\ \mathop{\mathrm{E}}_{\boldsymbol{\psi},\boldsymbol{\xi}\in\mathcal{S}_{n-1}} \exp(-\mathrm{i}\Omega\|\boldsymbol{x}-\boldsymbol{y}\|\boldsymbol{\psi}\cdot\boldsymbol{\xi}). \tag{10}$$

The angle ν between vectors $\boldsymbol{\psi}$ and $\boldsymbol{\xi}$ satisfies $\cos\nu = \boldsymbol{\psi}\cdot\boldsymbol{\xi}$. Without loss of generality, we can set $\boldsymbol{\xi} = (1, 0, 0, \ldots, 0)$ and rewrite the rotational spectrum as

$$S(\Omega) = \mathop{\mathrm{E}}_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{F}}\ H_n(\Omega\|\boldsymbol{x}-\boldsymbol{y}\|), \tag{11}$$

where the function $\mathrm{H}_n : \mathbb{R}\mapsto\mathbb{C}$ is defined as

$$\mathrm{H}_n(q) = \mathop{\mathrm{E}}_{\substack{\boldsymbol{\psi}\in\mathcal{S}_{n-1}\\ \psi_1=\cos\nu}} \exp(-\mathrm{i}q\cos\nu). \tag{12}$$

For $n = 1$, we obtain a degenerated rotation together with $\nu \in \{0, \pi\}$; therefore, the kernel function H_1 equals

$$\mathrm{H}_1(q) = \frac{\exp(-\mathrm{i}q)+\exp(\mathrm{i}q)}{2} = \cos q. \tag{13}$$

In case $n \geq 2$, we can express the kernel function using an integral formula:

$$\mathrm{H}_n(q) = \frac{\mathrm{I}_1(q)}{\mathrm{I}_2(q)} = \frac{\int_0^\pi \exp(-\mathrm{i}q\cos\nu)\sin^{n-2}\nu\,\mathrm{d}\nu}{\int_0^\pi \sin^{n-2}\nu\,\mathrm{d}\nu}. \tag{14}$$

The Poisson integral [17] formula for the Bessel function $\mathrm{J}_p(q)$ of the first kind in the form

$$\mathrm{J}_p(q) = \frac{\left(\frac{q}{2}\right)^p}{\Gamma\left(p+\frac{1}{2}\right)\sqrt{\pi}}\int_0^\pi \exp(-\mathrm{i}q\cos\nu)\sin^{2p}\nu\,\mathrm{d}\nu \tag{15}$$

allows the integral in the nominator to be rewritten as

$$\mathrm{I}_1(q) = \frac{\mathrm{J}_p(q)\Gamma\left(p+\frac{1}{2}\right)\sqrt{\pi}}{\left(\frac{q}{2}\right)^p}, \tag{16}$$

whereas the integral in the denominator is a limit case of the Poisson formula

$$\mathrm{I}_2(q) = \lim_{q\to 0}\frac{\mathrm{J}_p(q)\Gamma\left(p+\frac{1}{2}\right)\sqrt{\pi}}{\left(\frac{q}{2}\right)^p} = \frac{\Gamma\left(p+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(p+1)}. \tag{17}$$

For $p = \frac{n-2}{2}$, we obtain the final form of the kernel function expressed by the Bessel function $\mathrm{J}_p(q)$ as

$$\mathrm{H}_n(q) = \frac{2^{\frac{n-2}{2}}\cdot\Gamma\left(\frac{n}{2}\right)}{q^{\frac{n-2}{2}}}\mathrm{J}_{\frac{n-2}{2}}(q). \tag{18}$$

Applying $\mathrm{H}_n(q)$ for $n = 1$, we obtain $\mathrm{H}_1(q) = \cos q$ as a particular case, which extends the range of formula (18) to $n \in \mathbb{R}$. □

The rotation can be performed in any space whose dimension n is not less than the dimension m of the original space of $\mathcal{F}$. When the dimension of the rotation is greater than m, any vector $\boldsymbol{x} \in \mathcal{F}$ is completed, with the zeros for the remaining $n-m$ coordinates having a sufficient length. The most valuable result can be obtained in the case of rotation in an infinite-dimensional space.

Theorem 2. *The scaled limit case of the kernel function* H_n *is the Gaussian function, i.e.,*

$$\lim_{n\to\infty}\mathrm{H}_n(t\sqrt{n}) = \exp\left(-\frac{t^2}{2}\right). \tag{19}$$

Proof. For the investigation of the behaviour of the kernel function when $n \to \infty$, we use the Taylor expansion of $H_n(q)$ centred at $q_0 = 0$

$$\mathrm{H}_n(q) = \sum_{k=0}^{\infty}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+k)k!}\left(-\frac{q^2}{4}\right)^k, \tag{20}$$

and by using the substitution $q = t\sqrt{n}$, we can transform it into

$$\mathrm{H}_n(t\sqrt{n}) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{t^2}{2}\right)^k\frac{\Gamma(\frac{n}{2})n^k}{\Gamma(\frac{n}{2}+k)2^k}. \tag{21}$$

For every $k \in \mathbf{N}$, it holds that

$$\lim_{n\to\infty}\frac{\Gamma(\frac{n}{2})n^k}{\Gamma(\frac{n}{2}+k)2^k} = 1, \tag{22}$$