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Subharmonic bifurcations and chaotic motions for a class of inverted pendulum system

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ABSTRACT

Using both analytical and numerical methods, global dynamics including subharmonic bifurcations and chaotic motions for a class of inverted pendulum system are investigated in this paper. The expressions of the heteroclinic orbits and periodic orbits are obtained analytically. Chaos arising from heteroclinic intersections is studied with the Melnikov method. The critical curves separating the chaotic and non-chaotic regions are obtained. The conditions for subharmonic bifurcations are also obtained. It is proved that the system can be chaotically excited through finite subharmonic bifurcations. Some new dynamical phenomena are presented. Numerical simulations are given, which verify the analytical results.

1. Introduction

The inverted pendulum system has wide applications in precision instruments, robot control, missile intercept control system, spacecraft attitude control and so on. Therefore, it is of great significance to study nonlinear dynamics of this system. A lot of results on this subject have been obtained in the past two decades.

Bifurcations in the inverted pendulum system have been investigated by many researchers in the past years. Via the normal form theory, perturbation analysis and equivariant singularity theory, a reversible bifurcation analysis of the inverted pendulum was given by Broer et al. [1]. Using an approximating integrable normal form and equivariant singularity theory, Broer et al. [2] studied the qualitative dynamics of the Poincaré map corresponding to the central periodic solution and bifurcations of the inverted pendulum system. Ponce et al. [17] investigated bifurcations of an inverted pendulum with saturated Hamiltonian control laws. Periodic solutions and bifurcations in an impact inverted pendulum under impulsive excitation were investigated by Lenci and Rega [12]. It was found that the existence and the stability of the cycles depended on both classical (saddle-node and period-doubling) and non-classical bifurcations. With a quantitative theory together with numerical simulations, Butikov [4] studied the dynamic stabilization of an inverted pendulum. The dynamics of an inverted pendulum subjected to high-frequency excitation was studied by Yabuno et al. [22]. The stability of the stable equilibrium states under the effect of the tilt was discussed non-locally. An analogy of the bifurcation of the inverted pendulum to that of the buckling phenomenon was also presented. By using the method of multiple scales and numerical simulations, Yang et al. [23] investigated stability and Hopf bifurcation in an inverted pendulum with delayed feedback control. Via multiple delayed proportional gains, Boussaada et al. [3] studied a codimension-three triple zero bifurcation of an inverted pendulum on a cart moving horizontally. Using the centre manifold, Sieber and Krauskopf [18] investigated a codimension-three triple-zero eigenvalue bifurcation of an inverted pendulum with delayed feedback control. Using a center manifold reduction, Landry et al. [9] investigated local dynamics of an inverted pendulum with delayed feedback control. It was shown that the system undergoes a supercritical Hopf bifurcation at the critical delay. Employing a combination of analytical and numerical methods, the stability and bifurcations of two types of double impact periodic orbits for an inverted pendulum impacting between two rigid walls were studied by Shen and Du [19]. Especially, grazing bifurcations were presented there.

Chaotic motions of the inverted pendulum system have been also investigated in the past years. With numerical methods, Kim and Hu [8] studied bifurcations and transitions to chaos in an inverted pendulum. It was found that an infinite sequence of period-doubling bifurcations, leading to chaos, followed each destabilization of the inverted state. By using a Neyman–Pearson lemma like technique, Lenci [10] investigated the suppression of chaos by means of bounded excitations in an inverted pendulum. With experimental methods, Chen et al. [5] studied chaotic motions of a inverted pendulum system. It was found that the system may

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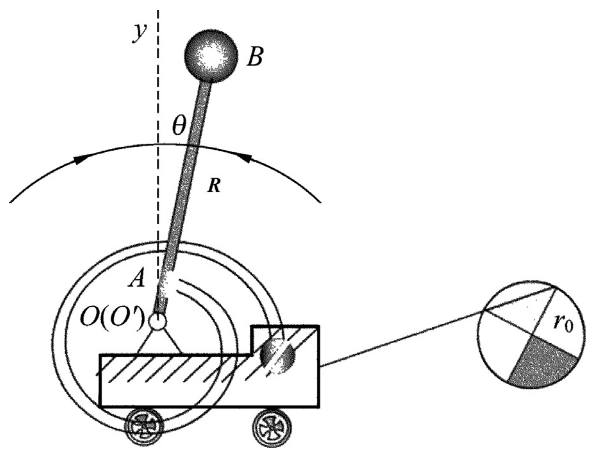

Fig. 1. The physical model of the inverted pendulum.

undergo chaos through period doubling bifurcations. Via a systematic numerical investigation method, Lenci et al. [11] studied the nonlinear dynamics of an inverted pendulum between lateral rebounding harriers. Three different families of considerably variable attractors-periodic, chaotic, and rest positions with subsequent chattering were found. Employing the harmonic balance method and Melnikov theory, nonlinear dynamics of an inverted pendulum driven by airflow was investigated by Nbendjo [14]. Horseshoes chaos may exist in this system. With the Melnikov method, homoclinic bifurcation for a nonlinear inverted pendulum impacting between two rigid walls under external quasi periodic excitations was analyzed by Gao and Du [7]. Smale horseshoe-type chaotic dynamics may occur in this system. Via Poincaré maps, Gandhi and Meena [6] studied chaotic dynamics of an inverted flexible pendulum with tip mass. By means of the normal form theory, the Melnikov method and numerical methods, Perez-Polo et al. [16] studied the stability and chaotic behavior of a plus integral plus derivative (PID) controlled inverted pendulum subjected to harmonic base excitations. It was shown that when the pendulum was close to the unstable pointing-up position, the PID parameters were changed and the chaotic motion was destroyed.

In this paper, global dynamics including subharmonic bifurcations and chaotic motions for a class of inverted pendulum system are investigated analytically with the subharmonic Melnikov method and Melnikov method, respectively. The mechanism and parameter conditions of chaotic motions are obtained rigorously. The critical curves separating the chaotic and non-chaotic regions are plotted. The chaotic feature on the system parameters is discussed in detail. The conditions for subharmonic bifurcations are also presented. It is proved that the system can be chaotically excited through finite subharmonic bifurcations. Numerical simulations verify the analytical results.

2. The dynamic model and analysis of the orbits for the inverted pendulum

Considering the model as in Fig. 1, assuming O is the center of the reciprocating motion for the system, choosing this point as the origin of the inertial coordinate system and the connection point of the inverted pendulum rod and the trolley as the origin of the non inertial coordinate system, then the dynamic equation of the ball in the non inertial coordinate system is [5]

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} = \sin\theta - \tilde{k}\theta - \gamma\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + A\cos\theta\cos\omega\tau \tag{1}$$

where $\tau = \sqrt{\frac{g}{R}}t$, $\tilde{k} = \frac{k'a^2}{mgR}$, $\omega = \omega_d/\sqrt{g/R}$, k' is the elastic coefficient of the spring, $\gamma = \frac{\tilde{\gamma}}{m\sqrt{g/R}}$, $\tilde{\gamma}$ is the damping coefficient, $A = r_0\omega_d^2/g = \frac{r_0}{R}\omega^2 \equiv f\omega^2$, r_0 is the excitation amplitude, R is the length of the pendulum, ω_d is the angular frequency of motor. Denoting $\theta = x$, then system (1) is written as follows :

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = y \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = \sin x - \tilde{k}x - \gamma y + f\omega^2\cos x\cos\omega\tau \end{cases} \tag{2}$$

Assuming the elastic coefficient of the spring together with the damping coefficient and the parameter f are small, setting $\tilde{k} = \varepsilon\bar{k}$, $\gamma = \varepsilon\bar{\gamma}$, $f = \varepsilon\bar{f}$, system (2) can be written as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = y \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = \sin x - \varepsilon\bar{k}x - \varepsilon\bar{\gamma}y + \varepsilon\bar{f}\omega^2\cos x\cos\omega\tau \end{cases} \tag{3}$$

When $\varepsilon = 0$, the unperturbed system of (3) is

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = y \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = \sin x \end{cases} \tag{4}$$

which is a planar Hamiltonian system with the Hamiltonian

$$H(x,y) = \frac{y^2}{2} + \cos(x) - 1 \equiv h \tag{5}$$

System (4) has saddles $(2l\pi, 0)(l = 0, \pm1, \pm2, \cdots)$, and centers $((2l+1)\pi, 0)(l = 0, \pm1, \pm2, \cdots)$. There exists a pair of heteroclinic orbits connecting $(2l\pi,\ 0)$ to $((2l+2)\pi, 0)$ when $h = 0$. Due to the periodic symmetry, we just need to consider the dynamic behaviors of the system on the interval $[0,\ 2\pi]$ of the x-axis. In this case, system (4) has two saddles $O(0,\ 0)$, $A(2\pi,\ 0)$, and one center $B(\pi,\ 0)$. To solve the expressions of the heteroclinic orbits, setting $h = 0$ in (5) one can obtain that

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = y = \pm\sqrt{2(1-\cos x)} \tag{6}$$

Separating variables for (6) and integrating, we can get

$$\tan\left(\frac{x}{4}\right) = e^{\pm\tau} \tag{7}$$

Therefore we can obtain the expressions of the homoclinic orbits as follows

$$\Gamma_+ : \begin{cases} x_+(\tau) = 4\arctan(e^{\tau}) \\ y_+(\tau) = 2\,\mathrm{sech}(\tau) \end{cases} \tag{8}$$

$$\Gamma_- : \begin{cases} x_-(\tau) = 4\arctan(e^{-\tau}) \\ y_-(\tau) = -2\,\mathrm{sech}(\tau) \end{cases} \tag{9}$$

There also exist a family of periodic orbits $\Omega_\pm(k)$ around the center B inside $\Gamma_\pm$ for $-2 < h < 0$. To solve the expressions of the periodic orbits, we rewrite (5) as follows

$$\cos x = (h+1) - \frac{y^2}{2} \tag{10}$$

consequently,

$$1 - \sin^2 x = \frac{y^4}{4} - (h+1)y^2 + (h+1)^2 \tag{11}$$

i.e.,

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{1}{C}(C^2 - y^2)(k'^2C^2 + k^2y^2) \tag{12}$$

where $k' = \sqrt{1-k^2}$. Comparing the coefficients of like powers, we can obtain that

$$C = \pm 2k, \quad h = 2(k^2 - 1) \tag{13}$$