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Time fractional quantum mechanics

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ABSTRACT

A time fractional quantum framework has been introduced into quantum mechanics. A new version of the space-time fractional Schrödinger equation has been launched. The introduced space-time fractional Schrödinger equation has a new scale parameter, which is a time fractional generalization of Planck's constant in quantum physics.

It has been shown that the presence of a fractional time derivative in the space-time fractional Schrödinger equation significantly impacts quantum mechanical fundamentals.

Time fractional quantum mechanical operators of coordinate, momentum and angular momentum were defined and their commutation relationships were established. The pseudo-Hamilton operator was introduced and its Hermicity has been proven.

Two new functions related to the Mittag-Leffler function have been introduced to solve the space-time fractional Schrödinger equation. Energy of a time fractional quantum system has been defined and calculated in terms of the newly introduced functions. It has been found that in the framework of time fractional quantum mechanics there are no stationary states, and the eigenvalues of the pseudo-Hamilton operator are not the energy levels of the time fractional quantum system.

A free particle solution to the space-time fractional Schrödinger equation was found. A free particle space-time fractional quantum mechanical kernel has been found and expressed in terms of the *H*-function. Renormalization properties of a free particle solution and the space-time fractional quantum kernel were established.

Some particular cases of time fractional quantum mechanics have been analyzed and discussed.

1. Introduction

In recent years the application of fractional calculus in quantum theory became a rapidly growing area. It was initiated by the discovery of fractional quantum mechanics (QM) [1–4]. The crucial manifestation of fractional QM is *fractional Schrödinger equation* . The fractional Schrödinger equation includes a spatial derivative of fractional order instead of the second order spatial derivative in the well-known Schrödinger equation. Thus, only the spatial derivative becomes fractional in the fractional Schrödinger equation, while the time derivative is the first order time derivative. Due to the presence of the first order time derivative in the fractional Schrödinger equation, fractional QM supports all QM fundamentals.

In [1,2] the path integral over Lévy-like quantum flights has been introduced, and the fractional Schrödinger equation was derived from the path integral over the Levy flights. The consideration presented in [2,4] is similar to the celebrated Feynman's derivation [5] of the well-know Schrödinger equation from the path integral over Brownian-like quantum paths.

To our best knowledge, the first attempt to elaborate on the quantum analogue of the Lévy flights probability distribution was the paper by Montroll [6]. Using the fact that a free particle quantum kernel satisfies the semi group property expressed by the chain equation, Montroll searched for a solution to the chain equation in the most general functional form. The solution obtained by Montroll is *the quantum analogue of the Lévy distribution*. A sequel to Montroll's paper was implemented by West in his seminal paper [7], where the fractional differential equation of motion for a free particle quantum Lévy kernel was found.

Inspired by the work of Laskin [1,2], Naber invented *time fractional Schrödinger equation* [8]. The time fractional Schrödinger equation involves the time derivative of fractional order instead of the first-order time derivative, while the spatial derivative is the second-order spatial derivative as it is in the well-known Schrödinger equation. To obtain the time fractional Schrödinger

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equation, Naber mapped the time fractional diffusion equation into the time fractional Schrödinger equation, similarly to the map between the well-known diffusion equation and the standard Schrödinger equation. The mapping implemented by Naber can be considered as a "fractional" generalization of the Wick rotation [9]. To get the time fractional Schrödinger equation, Naber implemented the Wick rotation in complex t -plane by rising the imaginary unit i to the same fractional power as the fractional order of the time derivative in the time fractional diffusion equation. The time fractional derivative in the time fractional Schrödinger equation is the Caputo fractional derivative [10]. Naber has found the exact solutions to the time fractional Schrödinger equation for a free particle and for a particle in a potential well [8].

Later on, Wang and Xu [11], and then Dong and Xu [12], combined both Laskin's equation and Naber's equation and came up with *space-time fractional Schrödinger equation*. The space-time fractional Schrödinger equation includes both spatial and temporal fractional derivatives. Wang and Xu found exact solutions to the space-time fractional Schrödinger equation for a free particle and for an infinite square potential well. Dong and Xu found the exact solution to the space-time fractional Schrödinger equation for a quantum particle in δ-potential well.

Here we introduce time fractional QM and develop its fundamentals. The wording "*time fractional quantum mechanics*" means that the time derivative in the fundamental quantum mechanical equations – Schrödinger equation and fractional Schrödinger equation, is substituted with a fractional time derivative. The time fractional derivative in our approach is the Caputo fractional derivative.

To introduce and develop time fractional QM we begin with our own version of the space-time fractional Schrödinger equation. Our space-time fractional Schrödinger equation involves two scale dimensional parameters, one of which can be considered as a time fractional generalization of the famous Planck's constant, while the other one can be interpreted as a time fractional generalization of the scale parameter emerging in fractional QM [1–4]. The time fractional generalization of Planck's constant is a fundamental dimensional parameter of time fractional QM, while the time fractional generalization of Laskin's scale parameter [1–4] plays a fundamental role in both time fractional QM and time fractional classical mechanics.

In addition to the above mentioned dimensional parameters, time fractional quantum mechanics involves two dimensionless fractality parameters α, $1 < \alpha \leq 2$ and β, $0 < \beta \leq 1$. Parameter α is the order of the spatial fractional quantum Riesz derivative [1] and β is the order of the time fractional derivative. In other words, α is responsible for modeling *spatial fractality*, while parameter β, which is the order of Caputo fractional derivative [10], is responsible for modeling *temporal fractality*.

Time fractional quantum mechanical operators of coordinate, momentum and angular momentum have been introduced and their commutation relationship has been established. The pseudo-Hamilton quantum mechanical operator has been introduced and its Hermiticity has been proven. The general solution to the space-time fractional Schrödinger equation was found in the case when the pseudo-Hamilton operator does not depend on time. Energy of a quantum system in the framework of time fractional QM was defined and calculated in terms of the Mittag-Leffler function. Two new functions associated with the Mittag-Leffler function have been launched and elaborated. These two new functions can be considered as a natural fractional generalization of the well-known trigonometric functions sine and cosine. A fractional generalization of the celebrated Euler equation was discovered. A free particle time fractional quantum kernel was calculated in terms of Fox's H -functions.

In the framework of time fractional QM at particular choices of fractality parameters α and β, we rediscovered the following fundamental quantum equations:

1. Schrödinger equation (Schrödinger equation [13]), $\alpha = 2$ and $\beta = 1$;
2. Fractional Schrödinger equation (Laskin equation [4]), $1 < \alpha \leq 2$ and $\beta = 1$;
3. Time fractional Schrödinger equation (Naber equation [8]), $\alpha = 2$ and $0 < \beta \leq 1$;
4. Space-time fractional Schrödinger equation (Wang and Xu [11] and Dong and Xu [12] equation), $1 < \alpha \leq 2$ and $0 < \beta \leq 1$.

1.1. Shortcomings of time fractional QM

While fractional QM [1–4] supports all QM fundamentals, time fractional QM violates the following fundamental quantum physics laws:

a. Quantum superposition law;
b. Unitarity of evolution operator;
c. Probability conservation law;
d. Existence of stationary energy levels of quantum system.

Time fractional quantum dynamics is governed by a pseudo-Hamilton operator instead of the Hamilton operator in standard QM and fractional QM. Eigenvalues of quantum pseudo-Hamilton operator are not the energy levels of a time fractional quantum system.

1.2. Benefits of time fractional QM

Despite of the above listed shortcomings, the developments in time fractional QM can be considered a newly emerged and attractive application of fractional calculus to quantum theory. Time fractional QM helps to understand the significance and importance of the fundamentals of QM such as Hamilton operator, unitarity of evolution operator, existence of stationary energy levels of quantum mechanical system, quantum superposition law, conservation of quantum probability, etc.

Besides, time fractional QM invokes new mathematical tools, which have never been used in quantum theory before.

From a stand point of QM the time fractional QM is an adequate, convenient mathematical framework well adjusted to study dissipative quantum systems interacting with environment [14–17].

1.3. The paper outline

The introduction presents a brief overview of benefits and shortcomings of time fractional QM.

In Section 2, we launched a new version of the space-time fractional Schrödinger equation. Our space-time fractional Schrödinger equation involves two scale dimensional parameters, one of which can be considered as a time fractional generalization of the famous Planck's constant, while the other one can be interpreted as a time fractional generalization of the scale parameter emerging in fractional QM. A 3D generalization of the space-time fractional Schrödinger equation has been developed. We also found the space-time fractional Schrödinger equation in momentum representation. The pseudo-Hamilton operator was introduced and its Hermiticity has been proven. The parity conservation law has been proven in the framework of time fractional QM.

In Section 3, the solution to the space-time fractional Schrödinger equation was obtained in the case when the pseudo-Hamilton operator does not depend on time. It was found that time fractional QM does not support a fundamental property of QM–conservation of quantum mechanical probability.