Chaos, Solitons and Fractals 000 (2017) 1–8

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals
Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Some properties of non-linear fractional stochastic heat equations on bounded domains

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ARTICLE INFO

Article history:
Received 15 December 2016
Revised 28 March 2017
Accepted 28 March 2017
Available online xxx

MSC:
Primary 60H15

Keywords:
Stochastic fractional PDEs
Large time behavior
Colored noise

ABSTRACT

We consider the following fractional stochastic partial differential equation on a bounded, open subset B of $\mathbb{R}^d$ for $d \geq 1$

$\partial_t u_t(x) = \mathcal{L}u_t(x) + \xi\sigma(u_t(x))\dot{F}(t,x),$

where ξ is a positive parameter and σ is a globally Lipschitz continuous function. The stochastic forcing term $\dot{F}(t,x)$ is white in time but possibly colored in space. The operator $\mathcal{L}$ is fractional Laplacian which is the infinitesimal generator of a symmetric α-stable Lévy process in $\mathbb{R}^d$. We study the behaviour of the solution with respect to the parameter ξ.

We show that under zero exterior boundary conditions, in the long run, the pth-moment of the solution grows exponentially fast for large values of ξ. However when ξ is very small we observe eventually an exponential decay of the pth-moment of this same solution.

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1. Introduction and main results

Stochastic Partial Differential Equations (SPDEs) have been used recently in many disciplines ranging from applied mathematics, statistical mechanics and theoretical physics to theoretical neuroscience, theory of complex chemical reactions (including polymer science), fluid dynamics and mathematical finance to quote only a few; see for example [9] and references therein.

In [8], the authors considered the following stochastic heat equation,

$$\partial_t u_t(x) = \mathcal{L}u_t(x) + \xi\sigma(u_t(x))\dot{F}(t,x), \tag{1.1}$$

where $\mathcal{L} = \Delta$ is the Dirichlet Laplacian on $B_R(0)$, the ball of radius R centered at the origin. Under some appropriate conditions, it was shown that the long time behaviour of the solution is dependent on the *noise level*, that is on the values of ξ. More precisely, it was shown that for large values of ξ, the moments of the solution grow exponentially with time while for small values of ξ, the moments decay exponentially. In this paper, we extend the results of Foondun and Nualart [8] and Xie [11] by taking $\mathcal{L}$ to be a non-local operator, the generator of a killed stable Lévy process, namely $\mathcal{L} := -\nu(-\Delta)^{\alpha/2}$ for $0 < \alpha \leq 2$ with zero exterior boundary conditions. The main results in this paper are Theorems 1.3 and 1.6 below. The fractional Laplacian is the infinitesimal generator of a symmetric α-stable Lévy process in $\mathbb{R}^d$ and can be written in the form

$$-(-\Delta)^{\frac{\alpha}{2}}u(x) = c\lim_{\varepsilon\downarrow 0}\int_{\{y\in\mathbb{R}^d:|y-x|>\varepsilon\}}(u(y)-u(x))\frac{\mathrm{d}y}{|y-x|^{d+\alpha}},$$

for some constant $c = c(\alpha, d)$. We also provide some clarification and simplification of the proofs in [8]. The difficulty in our paper lies in the fact that we need new estimates for the non-local operators. One cannot readily use analogous estimates for the Laplacian.

Non-local operators are becoming increasingly important due to their wide applicability for modeling purposes. The class of equations we study can for instance be used to model particles moving in a discontinuous fashion while being subject to some branching mechanism; see, for example, Walsh [10].

Throughout this paper, the initial condition u_0 is always assumed to be a non-negative bounded deterministic function such that for some set $K \subset B_R(0)$, the quantity

$$\int_K u_0(x)\,\mathrm{d}x$$

is strictly positive. The function σ will be subjected to the following condition.

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http://dx.doi.org/10.1016/j.chaos.2017.03.064
0960-0779/Published by Elsevier Ltd.

Assumption 1.1. The function σ is assumed to be a globally Lipschitz function satisfying

$$l_\sigma |x| \le |\sigma(x)| \le L_\sigma |x| \quad \text{for all} \quad x \in \mathbb{R},$$

for some positive constants l_σ and L_σ.

Following Walsh [10], we look at the mild solution of (1.1) satisfying the following integral equation,

$$u_t(x) = (\mathcal{G}_B u_0)_t(x) + \xi \int_{B_R(0)} \int_0^t p_B(t-s,x,y)\sigma(u_s(y))F(\mathrm{d}s,\mathrm{d}y), \tag{1.2}$$

where

$$(\mathcal{G}_B u_0)_t(x) = \int_{B_R(0)} u_0(y) p_B(t,x,y)\mathrm{d}y,$$

and $p_B(t, x, y)$ denotes the heat kernel of the stable Lévy process. It is the transition density of the stable Lévy process killed in the exterior of $B = B_R(0)$. More information about this kernel can be found in Section 2. When the driving noise is white in space and time, existence-uniqueness considerations impose the conditions that $d = 1$ and $1 < \alpha < 2$. When the noise term is not space-time white noise, it will be spatially correlated that is,

$$\mathbb{E}\dot{F}(s,\, x)\dot{F}(t,\, y) = \delta_0(t-s)f(x,y),$$

where the correlation function f satisfies the inequality $f(x,y) \le \tilde{f}(x-y)$, and $\tilde{f}$ is a locally integrable positive continuous function on $\mathbb{R}^d\backslash\{0\}$ satisfying the following Dalang type condition,

$$\int_{\mathbb{R}^d} \frac{\hat{\tilde{f}}(\xi)}{1+|\xi|^\alpha}\,\mathrm{d}\xi < \infty, \tag{1.3}$$

where $\hat{\tilde{f}}$ denotes the Fourier transform of $\tilde{f}$; see [4]. We will impose the following non-degeneracy condition on f,

Assumption 1.2. There exists a constant K_R such that

$$\inf_{x,y \in B_R(0)} f(x,y) \ge K_R.$$

The above conditions on the correlation function are quite mild. Examples of correlation functions satisfying Assumption 1.2 include the Riesz kernel, Cauchy kernels and many more: See, for example, [6] and [7].

Our first main result concerns Eq. (1.1) when the driving noise is space-time white noise which we denote by $\dot{W}$. In other words, we are looking at

$$\begin{cases} \partial_t u_t(x) = \mathcal{L}u_t(x) + \xi\sigma(u_t(x))\dot{W}(t,x), & x \in B_R(0), \quad t > 0 \\ u_t(x) = 0, \quad x \in B_R(0)^c. \end{cases} \tag{1.4}$$

Eq. (1.4) has a unique mild solution given in Eq. (1.2) with W replaced with F when σ is Lipschitz, $d = 1$ and $1 < \alpha < 2$. See [9,10] for more details on this.

Theorem 1.3. *Let $u_t(x)$ be the unique mild solution of Eq. (1.4), then there exists $\xi_0 > 0$ such that for all $\xi < \xi_0$ and $x \in B_R(0)$,*

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 < 0.$$

Fix $\varepsilon > 0$, then there exists $\xi_1 > 0$ such that for all $\xi > \xi_1$ and $x \in B_{R-\epsilon}(0)$,

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 < \infty.$$

Remark 1.4. It is not hard to see that $\xi_0 \le \xi_1$. Otherwise there will be an obvious contradiction in Theroem 1.3. We also give some estimates of ξ_0 and ξ_1 in Remark 3.1.

As in [8], we define the energy of the solution by the following quantity,

$$\mathcal{E}_t(\xi) = \sqrt{\mathbb{E}\|u_t\|^2_{L^2(B_R(0))}}. \tag{1.5}$$

The next corollary now follows easily from the above theorem.

Corollary 1.5. *With ξ_0 and ξ_1 as in Theorem 1.3, we have*

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \mathcal{E}_t(\xi) < 0 \quad \text{for all} \quad \xi < \xi_0$$

and

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathcal{E}_t(\xi) < \infty \quad \text{for all} \quad \xi > \xi_1.$$

Our next set of results concerns Eq. (1.1) with colored noise satisfying the conditions above. That is, we consider

$$\begin{cases} \partial_t u_t(x) = \mathcal{L}u_t(x) + \xi\sigma(u_t(x))\dot{F}(t,x), & x \in B_R(0), \quad t > 0 \\ u_t(x) = 0, \quad x \in B_R(0)^c. \end{cases} \tag{1.6}$$

Eq. (1.6) has a unique mild solution given by (1.2) when σ is Lipschitz and the spatial correlation function f satisfies the Dalang type condition in Eq. (1.3). See [6] for more details on this. Our second main result in this paper is the following theorem.

Theorem 1.6. *Assume that u_t is the unique mild solution to Eq. (1.6). Then there exists $\xi_2 > 0$ such that for all $\xi < \xi_2$ and $x \in B_R(0)$*

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 < 0.$$

Fix $\varepsilon > 0$, then there exists $\xi_3 > 0$ such that for all $\xi > \xi_3$ and $x \in B_{R-\varepsilon}(0)$,

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u_t(x)|^2 < \infty.$$

Remark 1.7. It is not hard to see that $\xi_2 \le \xi_3$. Otherwise there will be an obvious contradiction in Theroem 1.6. We can also give some estimates of ξ_2 and ξ_3 from the proof of Theorem 1.6 similar to the estimates in Remark 3.1.

We then have the following easy consequence.

Corollary 1.8. *Let ξ_2 and ξ_3 be as in Theorem 1.6, then*

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \mathcal{E}_t(\xi) < 0 \quad \text{for all} \quad \xi < \xi_2,$$

and

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathcal{E}_t(\xi) < \infty \quad \text{for all} \quad \xi > \xi_3.$$

We end this introduction with a plan of the article. In Section 2, we provide some estimates needed for the proofs of our main results. The proofs of our main Theorems are presented in Section 3. We then obtain extensions of the main results to higher moments in Section 4. Finally Section 5 contains some extensions of our main results to some other non-local operators instead of the fractional Laplacian. Throughout this paper, the letter c with or without subscript(s) will denote a constant whose value is not important and can vary from place to place.

2. Some estimates

We begin this section with some estimates on heat kernel of the Dirichlet fractional Laplacian in the ball $B := B_R(0)$. For more information on these, see Theorem 1.1 in [2] and references therein.

Please cite this article as: M. Foondun et al., Some properties of non-linear fractional stochastic heat equations on bounded domains, Chaos, Solitons and Fractals (2017), http://dx.doi.org/10.1016/j.chaos.2017.03.064