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Fractional Herglotz variational principles with generalized Caputo derivatives

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ABSTRACT

We obtain Euler–Lagrange equations, transversality conditions and a Noether-like theorem for Herglotz-type variational problems with Lagrangians depending on generalized fractional derivatives. As an application, we consider a damped harmonic oscillator with time-depending mass and elasticity, and arbitrary memory effects.

1. Introduction

Fractional variational principles and their applications is a subject under strong current research [3,19,20]. For classical fields with fractional derivatives, by using the fractional Lagrangian formulation, we can refer to [7]. A Hamiltonian approach to fractional problems of the calculus of variations is given in [25], where the Hamilton equations of motion are obtained in a manner similar to the one found in classical mechanics. In addition, classical fields with fractional derivatives are investigated using the Hamiltonian formalism [25]. A method for finding fractional Euler–Lagrange equations with Caputo derivatives, by making use of a fractional generalization of the classical Faá di Bruno formula, can be found in [5]. There the fractional Euler–Lagrange and Hamilton equations are obtained within the so called $1+1$ field formalism [5]. For discrete versions of fractional derivatives with a nonsingular Mittag–Leffler function see [1], where the properties of such fractional differences are studied and discrete integration by parts formulas proved in order to obtain Euler–Lagrange equations for discrete variational problems [1]. The readers interested in the discrete fractional calculus of variations are refereed to the pioneer work of Bastos et al. [8,9]. Here we are interested in the generalized continuous calculus of variations introduced by Herglotz.

The generalized variational principle firstly proposed by Gustav Herglotz in 1930 [14] gives a variational principle description of non-conservative systems even when the Lagrangian is autonomous [28,30]. It is essentially based on the following problem: find the trajectories $x(t)$, satisfying given boundary conditions, that extremize (minimize or maximize) the terminal value $z(b)$ of the functional z that satisfies the differential equation

$$\dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad t \in [a, b],$$

subject to the initial condition $z(a) = \gamma$. Herglotz proved that the necessary condition for a trajectory to be an extremizer of the generalized variational problem is to satisfy the generalized Euler–Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}} = 0.$$

The main physical motivation for the development of generalized variational methods behind the classical calculus of variations is linked to the inverse variational problem of classical mechanics in the cases where dissipation is not negligible [18]. In [2], Almeida

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and Malinowska have considered a fractional variational Herglotz principle, where fractionality stands in the dependence of the Lagrangian by the Caputo fractional derivative of the generalized variables. See also the more recent papers [33,34] on fractional Herglotz variational principles. In [33], new necessary conditions for higher-order generalized variational problems with time delay, which are semi-invariant under a group of transformations that depends on arbitrary functions, is obtained. Fractional variational problems of Herglotz type of variable order are investigated in [34], where necessary optimality conditions, described by fractional differential equations depending on a combined Caputo fractional derivative of variable order, are proved, both for one and several independent variables [34]. Using such results, it is possible to find, by using a variational approach, the equations of motion of a dissipative mechanical system with memory. This is useful, since in many classical cases, memory effects play a relevant role in systems with dissipation. A relevant example is given by the Basset memory force acting on a sphere rotating in a Stokes fluid. It is well-known that this force can be represented by means of Caputo derivatives of order 1/2 (see, e.g., [6] and references therein). However, a limit in this approach stands to the fact that the particular choice of the dependence of the Lagrangian by the Caputo fractional derivative of the generalized variable, implies that it describes equations of motion of systems with power-law memory kernels. On the other hand, in the framework of the fractional calculus of variations (a quite recent topic of research, started from the seminal investigations of Riewe [26] and then developed by many researchers, see for example the recent monographs [3,4,16,19,20]), Odzijewicz et al. [21,22] discusses the case of the Lagrangian depending on generalized Caputo-type derivatives with arbitrary completely monotonic kernels [19]. Here we consider a generalized calculus of variations in the sense of Herglotz with Lagrangians depending on generalized Caputo-type operators treated in [12,19,21]. Our aim is to find a general and adequate variational approach to describe mechanical systems with arbitrary memory forces.

The paper is organized as follows. In Section 2, we recall the necessary definitions and results from the generalized fractional variational calculus. Our results are then given in Sections 3–5: we prove in Section 3 necessary optimality conditions of Euler–Lagrange type (Theorem 3.1) and transversality conditions (Theorem 3.3) to the generalized fractional variational problem of Herglotz; we show in Section 4 how our approach can deal, in an elegant way, with dissipative dynamical systems with memory effects and time-varying mass and elasticity; and we obtain a generalized fractional Herglotz Noether theorem (Theorem 5.2) in Section 5. We end with Section 6 of conclusions and some directions of future work.

2. Preliminaries

In this section, we recall the main definitions of the generalized Riemann–Liouville and Caputo-like operators and their properties, according to the analysis of generalized fractional variational principles developed in [19,21]. For a general introduction to fractional differential operators and equations we refer to the classical encyclopedic book [27]. See also [24]. For an introduction to the fractional variational methods, and in particular integration by parts formulas for fractional integrals and derivatives, we refer to the monographs [4,20]. For computational and numerical aspects see [3,11,15].

Definition 2.1. The operator K_P^α is given by

$$K_P^\alpha[f](x) = K_P^\alpha[t \to f(t)](x)$$
$$= p\int_a^x k_\alpha(x,t)f(t)dt + q\int_x^b k_\alpha(t,x)f(t)dt,$$

where $P = \langle a, x, b, p, q\rangle$ is the parameter set, $x \in [a, b]$, $p, q \in \mathbb{R}$, and $k_\alpha(x, t)$ is a completely monotonic kernel.

For the sake of completeness, we should remark that similar generalizations of the Riemann–Liouville integrals have been considered in the framework of the fractional action-like variational approach (FALVA) [13,17]. On the other hand, a similar generalization is considered in a probabilistic framework in [35].

Theorem 2.2 (See [21, Theorem 3])**.** *Let $k_\alpha \in L_1([a, b])$ and*

$$k_\alpha(x,t) = k_\alpha(x-t).$$

Then, the operator $K_P^\alpha : L_1([a,b]) \to L_1([a,b])$ is a well-defined bounded and linear operator.

Definition 2.3. Let P be a given parameter set and $\alpha \in (0, 1)$. The operator $A^\alpha = D \circ K_P^{1-\alpha}$ is the generalized Riemann–Liouville derivative, where D stands for the conventional integer order derivative operator.

The corresponding generalized Caputo derivative is defined as $B_P^\alpha = K_P^{1-\alpha} \circ D$. A key-role in the following analysis is played by the following theorem that provides the integration by parts formula for the generalized operators defined before.

Theorem 2.4 (See [21, Theorem 11])**.** *Let $\alpha \in (0, 1)$ and $P = \langle a, x, b, p, q\rangle$. If $f, K_P^{1-\alpha}g \in AC([a, b])$, where $P^* = \langle a, x, b, q, p\rangle$ and $k_\alpha(\cdot)$ is a square-integrable function on $\Delta = [a, b] \times [a, b]$, then*

$$\int_a^b g(x)B_P^\alpha[f](x)dx = f(x)K_{P^*}^{1-\alpha}[g](x)\Big|_b^a - \int_a^b f(x)A_{P^*}^\alpha[g](x)dx.$$

3. Generalized fractional Herglotz variational principles

One of the main aims of this work is to prove generalized Euler–Lagrange equations related to the generalized fractional variational principle of Herglotz. In particular, the generalization is based on the fact that the Lagrangian depends on the generalized Caputo derivative B_P^α. As explained before, by using this approach, we will be able to find a variational approach to mechanical systems involving an arbitrary (suitable) memory kernel $k_\alpha(t)$. Therefore, let us consider the differential equation

$$\dot z(t) = L(t, x(t), B_P^\alpha[x](t), z(t)), \quad t \in [a, b], \tag{3.1}$$

with the initial condition $z(a) = z_a$. We moreover assume that

- $x(a) = x_a,\ x(b) = x_b,\ x_a, x_b \in \mathbb{R}^n$,
- $\alpha \in (0, 1)$,
- $x \in C^1([a,b], \mathbb{R}^n),\ B_P^\alpha[x] \in C^1([a,b], \mathbb{R}^n)$,
- the Lagrangian $L : [a, b] \times \mathbb{R}^{2n+1} \to \mathbb{R}$ is of class C^1 and the maps $t \to \lambda(t)\frac{\partial L}{\partial B_P^\alpha x_j}[x,z](t)$ exist and are continuous on $[a, b]$, where we use the notations

$$[x,z](t) := (t, x(t), B_P^\alpha[x](t), z(t)),$$
$$\lambda(t) = \exp\left(-\int_a^t \frac{\partial L}{\partial z}[x,z](\tau)d\tau\right).$$

The generalized fractional Herglotz variational principle is formulated as follows:

Let functional $z(t) = z[x; t]$ be given by the differential equation (3.1) and $\eta \in C^1([a, b], \mathbb{R})$ be an arbitrary function such that