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On disappearance of chaos in fractional systems

Amey S. Deshpande, Varsha Daftardar-Gejji*

Department of Mathematics, Savitribai Phule Pune University, Pune 411007, India

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ABSTRACT

In a seminal paper, Grigorenko and Grigorenko [15], numerically studied fractional order dynamical systems (FODS) of the form $D^{\alpha_i}x_i = f_i(x_1, x_2, x_3),\ 0 \le \alpha_i \le 1, (i = 1, 2, 3)$; and showed the existence of chaos in case of fractional Lorenz system when $\Sigma = \alpha_1 + \alpha_2 + \alpha_3 \le 3$. Since then voluminous numerical work has been done to explore various FODS, in this regard. It is now an established fact that Σ acts as a chaos controlling parameter.

In the present article we take a survey of present literature on chaotic behavior of fractional order dynamical systems. Further we numerically explore fractional Chen, Rössler, Bhalekar–Gejji, Lorenz and Liu systems and observe that chaos always disappear if $\Sigma \le 2$. The existing examples in the literature along with the systems that we have analyzed lead us to conjecture non-existence of chaos if $\Sigma \le 2$; which in some sense is a generalization of classical Poincaré–Bendixon theorem for FODS.

1. Introduction

Fractional order dynamical systems (FODS) have received attention after seminal work of Grigorenko and Grigorenko [15] in which they showed existence of chaos in fractional order Lorenz system. Since then various FODS have been investigated and critical value for the chaos to disappear (Σ_{cr}) has been calculated in each case.

Integer order Lorenz system was introduced by Lorenz in 1963 [21] in connection with hydrodynamics which he proved to be chaotic. Since then there are many other systems based on hydrodynamical models which have been shown to be chaotic [4,37,38]. Integer order chaotic Liu system was introduced by Liu et al. [20] in 2004. They showed that Liu system has complex dynamics which is different from dynamics of Lorenz system. Further Liu et al. realized this system with an electronic circuit. Rössler [27] introduced a chaotic system, which was simpler than the Lorenz system and has only one lobe. This system later found a number of applications in synchronization, cryptography and biology. Bhalekar and Daftardar-Gejji introduced a topologically inequivalent new system in 2011 [2] and proved existence of chaos in the same. The fractional version of this system is analyzed by Deshpande et al. recently [12].

In the present paper we take a survey of literature which deals with fractional versions of various systems and their chaotic behavior. We then explore the fractional version of Bhalekar–Gejji (BG) system, Lorenz system, Liu system, Chen system and Rössler system as a characteristic representatives of the set of all fractional order autonomous dynamical systems. We note that in each case when $\Sigma \le 2$, chaos always disappears, which leads us to conjecture that chaos cannot exist if $\Sigma \le 2$ for fractional dynamical systems.

Rest of the article is organized as follows. Section 2 defines preliminaries and notations used in the paper. Section 3 takes a survey of various fractional dynamical systems and their critical values below which chaos disappears. Section 4 describes details of the numerical method used for numerical simulations in this article. Section 5 presents numerical simulations and phase portraits for various systems. In Section 6 we conduct the stability analysis of the fractional systems. Section 7 lists the conclusions.

2. Preliminaries and notations

In this section, we introduce notations, definitions [19,24,26].

Definition 1 [26]. The Riemann–Liouville fractional integral of order $\alpha > 0$ of $f \in C^0$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{1}$$

* Corresponding author.
E-mail addresses: 2009asdeshpande@gmail.com (A.S. Deshpande), vsgejji@gmail.com, vsgejji@unipune.ac.in (V. Daftardar-Gejji).

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Definition 2 [26]**.** The Riemann–Liouville derivative of order $\alpha \in [k-1, k),\ k \in \mathbb{N}$ of $f \in C$ is defined as

$$
\begin{aligned}
{}_{RL}D^{\alpha} f(t) &= D^k I^{k-\alpha} f(t) \\
&= \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dt^k} \int_0^t (t-\tau)^{k-\alpha-1} f(\tau)\, d\tau.
\end{aligned} \tag{2}
$$

Definition 3 [26]**.** The Caputo derivative of order $\alpha \in (k-1, k],\ k \in \mathbb{N}$ of $f \in C^k$ is defined as

$$
\begin{aligned}
D^{\alpha} f(t) &= I^{k-\alpha} f^{(k)}(t) \\
&= \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-\tau)^{k-\alpha-1} f^{(k)}(\tau)\, d\tau, \quad \alpha \in (k-1, k),
\end{aligned} \tag{3}
$$

$$
D^{\alpha} f(t) = f^{(k)}(t), \ \alpha = k. \tag{4}
$$

In this article we deal with Caputo derivative of order $\alpha, 0 < \alpha \le 1$.

Definition 4 [19]**.** Let $\alpha = (\alpha_1, \alpha_2, \alpha_3),\ 0 < \alpha_i \le 1,\ (i = 1, 2, 3)$. For $x(t) \in \mathbb{R}^3$ and $f = (f_1, f_2, f_3),\quad f_i \in C^1,\ (i = 1, 2, 3)$, fractional dynamical system is defined as

$$
D^{\alpha} x(t) = f(x(t)),\ x(0) = x_0, \ \text{ where } \ D^{\alpha} = (D^{\alpha_1}, D^{\alpha_2}, D^{\alpha_3}). \tag{5}
$$

If $\alpha_1 = \alpha_2 = \alpha_3$ then the system (5) is called as *commensurate* system and *incommensurate* system otherwise [31]. For $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the system (5) reduces to integer order dynamical system.

Let Σ denote the sum of all fractional orders of (5) *i.e.* $\Sigma = \alpha_1 + \alpha_2 + \alpha_3$.

Definition 5. The critical value (Σ_{cr}) is defined as the largest value of Σ, such that for the $\Sigma \le \Sigma_{cr}$, the system of (5) is not chaotic.

Definition 6. The point $x^* = (x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3$ is called as equilibrium point of the system (5), if $f(x^*) = 0$.

2.1. Stability analysis of the commensurate fractional systems

Consider the system (5), with $\alpha_1 = \alpha_2 = \alpha_3 \in (0, 1]$.

The equilibrium point is called as hyperbolic equilibrium point if $|\arg(\lambda)| \ne \frac{\pi\alpha}{2}$, for any eigenvalue λ of the matrix $J = Df(x^*)$ [23]. Assume x^* is an equilibrium point of the system (5). Let $\xi = x - x^* \in \mathbb{R}^3$, then [19]

$$
\begin{aligned}
D^{\alpha}\xi &= D^{\alpha}(x - x^*) \\
&= D^{\alpha} x \\
&= f(\xi + x^*) \\
&= f(x^*) + Df(x^*)\xi + \cdots
\end{aligned}
$$

Thus we get,

$$
D^{\alpha}\xi = J\xi, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3),\ \alpha_1 = \alpha_2 = \alpha_3 \in (0, 1]. \tag{6}
$$

The eigenvalues of the matrix J determine the stability properties of the system about the equilibrium point x^* [19]. The stability criteria is due to Matignon [23] which states: If x^* is a hyperbolic equilibrium point then the trajectories of the system (6) are asymptotically stable if and only if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$, for every eigenvalue λ of J.

2.2. Stability analysis of the incommensurate fractional order systems

Consider the system (5), with $\alpha_i = \dfrac{v_i}{u_i},\quad (v_i, u_i) = 1,\quad u_i, v_i \in \mathbb{N},\ i = 1, 2, 3$, and $\alpha_i \in (0, 1)$. Define $M = \mathrm{LCM}(v_1, v_2, v_3)$. Let x^* denote an equilibrium point of the system (5), and $\xi = x - x^*$. Using the similar analysis as in previous subsection,

$$
D^{\alpha_i}\xi_i \approx \xi_1 \frac{\partial}{\partial x_1} f_i(x^*) + \xi_2 \frac{\partial}{\partial x_2} f_i(x^*) + \xi_3 \frac{\partial}{\partial x_3} f_i(x^*), \quad 1 \le i \le 3, \tag{7}
$$

is equivalent to

$$
D^{\alpha}\xi = J\xi, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3),\ \alpha_i \in (0, 1],\ i = 1, 2, 3. \tag{8}
$$

where J denotes the Jacobian matrix evaluated at x^*.

Define $\Delta(\lambda) = \mathrm{diag}([\lambda^{M\alpha_1}\ \lambda^{M\alpha_2}\ \lambda^{M\alpha_3}]) - J$.

Then the solution of the linear system (8) is asymptotically stable, if all the roots of the equation $\Delta(\lambda) = 0$, satisfy the condition [11,31]

$$
\frac{\pi}{2M} - \min_i (|\arg(\lambda_i)|) < 0. \tag{9}
$$

Thus the equilibrium point x^* is stable, if the condition (9) is satisfied. The $\frac{\pi}{2M} - \min(|\arg(\lambda_i)|)$ is defined as the instability measure of the fractional order systems (IMFOS). Tavazoei et al. [31] show that the necessary condition for a fractional incommensurate system to be chaotic is

$$
\mathrm{IMFOS} \ge 0. \tag{10}
$$

It should be noted that the condition (10) is not sufficient for concluding existence of chaos [31].

3. Survey of fractional dynamical systems and chaos

In this section we take a detailed survey of the research articles which analyze various fractional systems for chaotic behavior and their critical values.

Grigorenko and Grigorenko [15] in their seminal article explored fractional order Lorenz system and found that the critical value for chaos to disappear $\Sigma_{cr} = 2.91$ [15]. Fractional version of Chua system was simulated by Hartley et al. and they found Σ_{cr} to be 2.7 for this system [16]. Ahmad and Sprott analyzed fractional version of ‘jerk’ model over wide set of parameter values. From this analysis they found the critical value to be 2.1 for this system [1]. Fractional version of Rössler system was analyzed by Li and Chen and Σ_{cr} for this system is shown to be 2.4 [17]. Some of the systems such as fractional Arneodo system [22], fractional Lü system [10], fractional Chen system [18] were analyzed for chaos by using frequency domain approximation method. Some of these systems were reported to show chaotic behavior for very low orders of fractional derivatives. However frequency domain approximation method was proven to be unreliable and the Σ_{cr} values found in these articles were shown to be incorrect by Tavazoei and Haeri [32]. Further Tavazoei and Haeri showed $\Sigma_{cr} = 2.43$ for fractional Lü system while $\Sigma_{cr} = 2.4$ for fractional Chen system in the same article. Fractional Liu system was explored for chaos by Wang and Wang [34] and critical value for chaos to disappear was found to be 2.544 [34]. A “new system” was proposed and analyzed for chaos by Sheu et al. and Σ_{cr} was found to be 2.43 [28]. Sheu et al. also numerically explored Newton-Leipnik system and found $\Sigma_{cr} = 2.82$ [29]. Fractional version of Financial system was explored by Chen for chaos and Σ_{cr} was found to be 2.55 for commensurate case and 2.35 for incommensurate system [6]. A Unified system was analyzed by Wu et al. and Σ_{cr} was shown to be 2.76 [36]. A new Liu system was investigated by Daftardar-Gejji and Bhalekar [7] and Σ_{cr} was shown to be 2.76 for commensurate while 2.60 for incommensurate case [7]. A fractional version of simplified Lorenz system proposed by Sun et al. was analyzed for bifurcations and chaos by Sun and Sprott [30]. They found that minimum value of fractional order for which chaos disappears as 2.62. Fractional version of Volta’s system was shown to be chaotic by Petráš and Σ_{cr} was shown to be 2.9 [25]. Fractional Genesio-Tesi system was analyzed by Faieghi et al. and they found Σ_{cr} value to be 2.79 [14]. Fractional version of Bloch equation was put to simulation and shown to be transiently chaotic below certain values of fractional order by Bhalekar et al. [3]. They found that $\Sigma_{cr} = 2.97$ for this system [3]. The fractional version of Bhalekar–Gejji system was analyzed for bifurcations and chaotic behavior over wide

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