Chaos, Solitons and Fractals 000 (2017) 1–8

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals
Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Mellin integral transform approach to analyze the multidimensional diffusion-wave equations

Lyubomir Boyadjiev[a], Yuri Luchko[b,*]

[a] Department of Mathematics, Faculty of Science, Kuwait University, 13060–Safat, Kuwait
[b] Department of Mathematics, Physics, and Chemistry, Beuth University of Applied Sciences Berlin, Luxemburger Str. 10, 13353 – Berlin, Germany

ARTICLE INFO

Article history:
Received 27 January 2017
Revised 19 March 2017
Accepted 24 March 2017
Available online xxx

MSC:
26A33
35C05
35E05
35L05
45K05
60E99

Keywords:
Time- and space-fractional multidimensional diffusion-wave equation
Fundamental solution
Mellin integral transform
Mellin-Barnes integral representation
Probability density function

ABSTRACT

In this paper, a family of the multidimensional time- and space-fractional diffusion-wave equations with the Caputo time-fractional derivative of the order $\beta, 0 < \beta \leqslant 2$ and the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ with $1 < \alpha \leqslant 2$ is considered. A representation of the first fundamental solution to this equation is deduced in form of a Mellin–Barnes integral by employing the technique of the Mellin integral transform. The Mellin–Barnes representation is used to derive some new identities for the fundamental solutions in different dimensions and to identify already known and some new particular cases of the fundamental solution that have especially simple closed form.

1. Introduction

The partial differential equations of fractional order are nowadays a topic under very intensive development. The spectrum of papers devoted to this theme is mainly threefold: fractional partial differential equations as models for real processes and systems, analysis of physical and probabilistic properties of their solutions, and, finally, mathematical and numerical analysis of these equations and their solutions. This paper falls into the third category of papers mentioned above, so that we are not going to discuss any applications of the equations we deal with (even if we are aware of some applications and recognize their importance, see e.g [27]., [28]).

In its turn, the variety of publications devoted to mathematical analysis of the fractional partial differential equations is immense, too. It can be explained among other things by the fact, that instead of a unique notion of a derivative of integer order, there are many different definitions of the fractional derivatives (in the sense of Riemann–Liouville, Weyl, Caputo, Riesz, Riesz–Feller, Grünwald–Letnikov, etc.) and differential equations with all these derivatives can be considered (at least from the mathematical viewpoint). Furthermore, initial and boundary conditions for the fractional partial differential equations can be posed both in local and non-local forms (say, in terms of the fractional derivatives and integrals) that enlarge the quantity of potentially interesting problems even more.

Most of papers devoted to different aspects of the fractional partial differential equations deal with the spatial one-dimensional problems. The literature dedicated to the multidimensional fractional partial differential equations is still very restricted and mostly treats the time-fractional partial differential equations. The fundamental solution to the Cauchy problem for the multidimensional time-fractional diffusion equation with the time-fractional derivative of the order $\alpha, 0 < \alpha \leqslant 2$ and the Laplace operator was derived independently in [13] and [31] in terms of the Fox *H*-function. Moreover, uniqueness and existence of solutions to the Cauchy problem with initial conditions from appropriate functional spaces were investigated in [13], too. In [4] these results were extended to the class of the multidimensional time-fractional

* Corresponding author.
E-mail addresses: boyadjievl@yahoo.com (L. Boyadjiev), luchko@beuth-hochschule.de (Y. Luchko).

http://dx.doi.org/10.1016/j.chaos.2017.03.050
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diffusion equations with a uniformly elliptic operator with variable coefficients acting in the spatial variables and with the time-fractional derivative of the order $\alpha, 0 < \alpha < 1$ and in [14] these equations with the time-fractional derivative of order $\alpha, 1 < \alpha < 2$ were investigated. In [12], the Green function and propagators for the multidimensional time-fractional diffusion-wave equation were derived in integral form. Based on the integral representations of the solutions, their physical properties were discussed in this paper, too. A representation of the fundamental solution to the multidimensional time-fractional diffusion equation was derived in [27] in terms of the multidimensional Gaussian density function. In the very recent paper [7], a representation of the fundamental solution to the multidimensional time-fractional diffusion-wave equation in terms of the Fox H-function was rediscovered. This representation was employed for derivation of the series representations of the fundamental solution and for calculation of its fractional moments.

Recently, a series of papers regarding the multidimensional time-fractional diffusion-wave equation on bounded spatial domains was published. In [17], some uniqueness and existence results for the solutions of the initial-boundary-value problems for the generalized time-fractional diffusion equation with the Caputo fractional derivative were given. To establish uniqueness of solution, a maximum principle for the generalized time-fractional diffusion equation proved in [16] was employed. In [18] the case of the initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation with the Caputo derivative was considered. Let us also mention the papers [1] and [2] devoted to the initial-boundary-value problems for the time-fractional diffusion equation with the Riemann–Liouville fractional derivative and the paper [24], where the initial-boundary-value problems for a general time-fractional diffusion equation which generalizes the single- and the multi-term time-fractional diffusion equations as well as the time-fractional diffusion equation of the distributed order were considered.

The literature devoted to the multidimensional space- and time-space-fractional partial differential equations is even more restricted. Especially worth to mention are the papers [10] and [11], where the Green functions and propagators for the multidimensional space and space-time-fractional diffusion equations were derived in integral form and investigated from the physical viewpoint. In [20], a multi-dimensional fractional wave equation that contains the time- and space-fractional derivatives of the same order $\alpha, 1 \leqslant \alpha \leqslant 2$ was introduced and investigated. In particular, some new integral representations of its fundamental solution were deduced and employed for derivation of its mathematical and physical properties. In the one- and three-dimensional cases, the fundamental solution to the fractional wave equation has a nice closed form in terms of some elementary functions. In [22], the case of the two-dimensional space- and time-fractional diffusion equation that contains the Caputo time-fractional derivative of order $\alpha/2$ and the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ with $0 < \alpha < 2$ was considered. The fundamental solution to this equation was shown to be a two-dimensional probability density function that can be expressed in explicit form in terms of the Mittag–Leffler function. Otherwise, the form of the fundamental solution to the one- and three-dimensional space- and time-fractional diffusion equation is complicated and cannot be expressed in form of some elementary and known classical special functions (see [3]). It is worth mentioning that in the papers [11], [12], and [31], an interpretation of the fundamental solutions to the fractional diffusion equations as a probability density function was given for the orders of the time-fractional derivative less than or equal to one. In this paper, the order of the fractional derivative is supposed to be from the interval (0, 2), but we do not treat the conditions for the non-negativity of the fundamental solution to the multidimensional diffusion-wave equation that remains an open problem.

These results lead us to the following question that seems to be non-answered yet: How to determine all particular cases of the multidimensional space- and time-fractional diffusion-wave equation with the Caputo time-fractional derivative of the order β and the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ that have a closed form in terms of the elementary and simple special functions. To answer this question, in this paper the technique of the Mellin integral transform (see [23] for applications of the Mellin integral transform in Fractional Calculus), the Mellin–Banes integral representations, and the properties of the Euler Gamma-functions are employed. This leads to a series of known and new particular cases of the fundamental solutions and to the formulas that connect the fundamental solutions for different dimensions. Let us note that a thorough analysis of the one-dimensional space- and time-fractional diffusion-wave equation has been done in [25].

The rest of the paper is organized as follows: In the second section, a problem formulation and some basic definitions are presented. The third section is devoted to derivation of the Mellin–Barnes integral representations of the fundamental solution to the multidimensional space- and time-fractional diffusion-wave equation with the Caputo time-fractional derivative of the order β and the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$. In the fourth section, these representations are employed for derivation of some identities that connect the fundamental solutions for different dimensions and especially simple closed form formulas for the fundamental solutions. Open questions for further research are listed in the last section.

2. Problem formulation

In this paper, we deal with the multidimensional space- and time-fractional diffusion-wave equation in the following form:

$$D_t^\beta u(\mathrm{x},t) = -(-\Delta)^{\frac{\alpha}{2}} u(\mathrm{x},t), \quad \mathrm{x} \in \mathbb{R}^n,\ t > 0,$$
$$1 < \alpha \leqslant 2,\ 0 < \beta \leqslant 2, \tag{1}$$

where $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian and D_t^β is the Caputo time-fractional derivative of the order β. The Caputo time-fractional derivative of order $\beta > 0$ is given by the formula

$$D_t^\beta u(\mathrm{x},t) = \left(I_t^{n-\beta} \frac{\partial^n u}{\partial t^n}\right)(t), \quad n-1 < \beta \leqslant n,\ n \in \mathbb{N}, \tag{2}$$

where I_t^γ is the Riemann–Liouville fractional integral that is defined by

$$(I_t^\gamma u)(t) = \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^t (t-\tau)^{\gamma-1} u(\mathrm{x},\tau)\,d\tau & \text{for } \gamma > 0, \\ u(\mathrm{x},t) & \text{for } \gamma = 0. \end{cases}$$

For a sufficiently well-behaved function $f : \mathbb{R}^n \to \mathbb{R}$, the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is defined as a pseudo-differential operator with the symbol $|\kappa|^\alpha$ ([29,30]):

$$\left(\mathcal{F}\,(-\Delta)^{\frac{\alpha}{2}} f\right)(\kappa) = |\kappa|^\alpha (\mathcal{F} f)(\kappa), \tag{3}$$

where $(\mathcal{F} f)(\kappa)$ is the Fourier transform of a function f at the point $\kappa \in \mathbb{R}^n$ defined by

$$(\mathcal{F} f)(\kappa) = \hat{f}(\kappa) = \int_{\mathbb{R}^n} e^{i\kappa\cdot\mathrm{x}} f(\mathrm{x})\,d\mathrm{x}. \tag{4}$$

For $0 < \alpha < m,\ m \in \mathbb{N}$ and $x \in \mathbb{R}^n$, the fractional Laplacian can be also represented as a hypersingular integral ([30]):

$$(-\Delta)^{\frac{\alpha}{2}} f(\mathrm{x}) = \frac{1}{d_{n,m}(\alpha)} \int_{\mathbb{R}^n} \frac{\left(\Delta_{\mathrm{h}}^m f\right)(\mathrm{x})}{|\mathrm{h}|^{n+\alpha}}\,d\mathrm{h} \tag{5}$$

with the suitably defined finite differences operator $\left(\Delta_{\mathrm{h}}^m f\right)(\mathrm{x})$ and the normalization constant $d_{n,m}(\alpha)$. The operator $\left(\Delta_{\mathrm{h}}^m f\right)(\mathrm{x})$ can be