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Existence results for coupled nonlinear fractional differential equations equipped with nonlocal coupled flux and multi-point boundary conditions

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ABSTRACT

We introduce a new concept of coupled flux conditions and unify it with nonlocal coupled strip and multi-point boundary conditions. Equipped with the unified boundary conditions, a nonlinear coupled system of Liouville-Caputo type fractional differential equations is studied. Existence and uniqueness results for the given boundary value problem are obtained by applying Banach's fixed point theorem and Leray–Schauder alternative, and are well illustrated with the aid of examples. Our work is not only new in the given configuration but also yields several new results as its special cases.

1. Introduction

In this paper, we discuss the existence of solutions for a nonlinear coupled system of Liouville-Caputo type fractional differential equations:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t), y(t), {}^cD^{\sigma_1} y(t)), \\ \qquad 2 < q \le 3,\ 1 < \sigma_1 < 2,\ t \in [0,1], \\ {}^cD^p y(t) = g(t, x(t), {}^cD^{\sigma_2} x(t), y(t)), \\ \qquad 2 < p \le 3,\ 1 < \sigma_2 < 2,\ t \in [0,1], \end{cases} \tag{1.1}$$

equipped with coupled flux conditions unified with nonlocal strip and multi-point boundary conditions:

$$\begin{cases} x(0) = \psi_1(y),\ x'(0) = e_1 y'(w_1), \\ \quad x(1) = a_1 \int_0^{\xi} y(s)ds + b_1 \sum_{i=1}^{m-2} \alpha_i y(\eta_i), \\ y(0) = \psi_2(x),\ y'(0) = e_2 x'(w_2), \\ \quad y(1) = a_2 \int_0^{\xi} x(s)ds + b_2 \sum_{i=1}^{m-2} \beta_i x(\eta_i), \\ 0 < w_1 < w_2 < \xi < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, \end{cases} \tag{1.2}$$

where ${}^cD^{\varsigma}$ denote the Liouville-Caputo fractional derivative of order ς with $\varsigma = p, q, \sigma_1, \sigma_2, f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\psi_1, \psi_2 : C([0,1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, a_1, a_2, b_1, b_2, e_1 and e_2 are real constants and α_i, β_i, $i = 1, 2, \ldots, m-2$, are positive real constants.

Here we emphasize that the main objective of the present work is to introduce a new set of fully coupled nonlocal flux, strip and multi-point boundary conditions and solve a coupled system of Liouville-Caputo type nonlinear fractional differential equations equipped with these conditions. The proposed problem is of quite a general nature as it covers several special cases. Thus the obtained results will be a useful and novel contribution to the existing literature on fractional order boundary value problems.

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The study of coupled systems of fractional-order differential equations has gained considerable attention as such systems appear in the mathematical modeling of many real world problems. Examples include anomalous diffusion [1], disease models [2–5], synchronization of chaotic systems [6,7], ecological models [8], etc. For some recent results on coupled systems of fractional-order differential equations, we refer the reader to a series of papers [9–15].

The popularity of fractional calculus tools in the mathematical modeling of many processes and phenomena is quite eminent. It has been mainly due to the fact that fractional-order operators are nonlocal in nature in contrast to integer-order operators and are capable of tracing the past effects of the involved phenomena. For examples and details, see [16–21].

The topic of fractional-order boundary value problems has been addressed by many authors and a significant development on the subject can be witnessed in the recent literature. For some recent works, we refer the reader to [22–33] and the references cited therein.

The rest of the paper is organized as follows. In Section 2, we recall some definitions for quick reference and prove an auxiliary lemma that lays the foundation for solving the given problem. Main results are presented in Section 3, while Section 4 contains concluding remarks and some interesting observation.

2. Preliminaries

First of all, we recall some basic definitions of fractional calculus.

Definition 2.1. The fractional integral of order r with the lower limit zero for a function $f : [0, \infty) \to R$ is defined as

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{f(s)}{(t-s)^{1-r}} ds, \quad t > 0, \quad r > 0,$$

provided the right hand-side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$.

Definition 2.2. The Riemann–Liouville fractional derivative of order $r > 0,\ n-1 < r < n,\ n \in N,$ is defined as

$$D_{0+}^r f(t) = \frac{1}{\Gamma(n-r)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-r-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3. The Caputo derivative of order r for a function f: $[0, \infty) \to R$ can be written as

$${}^cD^r f(t) = D_{0+}^r \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n-1 < r < n.$$

Remark 2.4. If $f(t) \in C^n[0, \infty)$, then

$${}^cD^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{r+1-n}} ds = I^{n-r} f^{(n)}(t),\ t > 0,\ n-1 < q < n.$$

Now we present an auxiliary lemma which plays a key role in the sequel.

Lemma 2.5. *For $\widehat{f}, \widehat{g} \in C[0,1]$, the solution of the linear system of fractional differential equations*

$$
\begin{aligned}
&{}^cD^q x(t) = \widehat{f}(t), \quad 2 < q \le 3,\ t \in [0,1],\\
&{}^cD^p y(t) = \widehat{g}(t), \quad 2 < p \le 3,\ t \in [0,1],
\end{aligned}
\qquad (2.1)
$$

supplemented with the boundary conditions (1.2) is equivalent to the system of integral equations

$$
\begin{aligned}
x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \widehat{f}(s) ds + \psi_1(y)[1 - K_3(t) + \chi_2 K_4(t)] + \psi_2(x)[\chi_1 K_3(t) - K_4(t)]\\
& + K_1(t) e_1 \int_0^{w_1} \frac{(w_1-s)^{p-2}}{\Gamma(p-1)} \widehat{g}(s) ds + K_2(t) e_2 \int_0^{w_2} \frac{(w_2-s)^{q-2}}{\Gamma(q-1)} \widehat{f}(s) ds\\
& + K_3(t) \Big[a_1 \int_0^{\xi} \frac{(\xi-s)^p}{\Gamma(p+1)} \widehat{g}(s) ds + b_1 \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} \frac{(\eta_i-s)^{p-1}}{\Gamma(p)} \widehat{g}(s) ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \widehat{f}(s) ds \Big]\\
& + K_4(t) \Big[a_2 \int_0^{\xi} \frac{(\xi-s)^q}{\Gamma(q+1)} \widehat{f}(s) ds + b_2 \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} \frac{(\eta_i-s)^{q-1}}{\Gamma(q)} \widehat{f}(s) ds - \int_0^1 \frac{(1-s)^{p-1}}{\Gamma(p)} \widehat{g}(s) ds \Big], \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
y(t) = {} & \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} \widehat{g}(s) ds + \psi_2(x)[1 + \chi_1 K_7(t) - K_8(t)] + \psi_1(y)[\chi_2 K_8(t) - K_7(t)]\\
& + K_5(t) e_1 \int_0^{w_1} \frac{(w_1-s)^{p-2}}{\Gamma(p-1)} \widehat{g}(s) ds + K_6(t) e_2 \int_0^{w_2} \frac{(w_2-s)^{q-2}}{\Gamma(q-1)} \widehat{f}(s) ds
\end{aligned}
$$