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Fractional derivatives on cosmic scales

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ABSTRACT

Almost since the very discovery of cosmic rays, calculations of their propagation in the Galaxy have been based on the use of the local diffusion model. It is quite acceptable for modeling of the Brownian motion because displacements of the Brownian tracer are mutually independent in length and direction. However, some features of this model are incompatible with the real behavior of cosmic rays: the path of the tracer between any two points of its trajectory is infinite, the local velocity is infinite and the front from a local pulse source is absent (after a moment, the particle can be observed arbitrarily far away from its original place). In this article, we describe the current state of a new model of cosmic ray transport, being free from these imperfections. It was formulated in 2010 and is still in progress under the title NoRD (Nonlocal Relativistic Diffusion) model. Two crucial ideas underlie this approach: taking into account correlations in space-time increments and using the material derivative of a fractional order. First of them ensures the relativistic speed-limitation whereas the second one reflects the influence of interstellar medium turbulence. The numerical calculation results demonstrated in this paper do speak well for the NoRD-model as compared with the traditional one based on integer-order operators.

1. Introduction

The transport of cosmic rays (CRs) in the interstellar medium remains an important problem in astrophysics. The highest-energy CRs are the most energetic of any particles ever observed in nature: they have a hundred million times more energy than the particles produced in the most powerful particle accelerator at the Earth. The remarkable feature of CRs is the power law spectrum $N(E) \propto E^{-\gamma}$ (with $\gamma \approx 2.8 - 3.2$) extending over 11 orders of particle energy.

The first explanation (or rather prediction) of the power law of primary CRs spectrum given by E. Fermi was purely phenomenological: it was based on the exponential growth of energy with time and exponential distribution of ages of observed particles. Both of them ignored real space-time distribution of sources, highly inhomogeneous structure of the interstellar medium and boundedness of the Galaxy. A few years later, the CR propagation was interpreted as some kind of Brownian motion (Bm) and began to be calculated in terms of the ordinary (normal) diffusion theory [1]. It is easy to understand the origin of this model: Fermi wrote about random walk of CR particles among magnetic clouds chaotically distributed in interstellar space, and Bm was the most expressive image of such a process.

However, the random process, underlying Bm, is of Markovian type, that is, its consecutive increments are mutually independent. This means that the environment consists of entities distributed in interstellar space uniformly and independently of each other. Only under this condition, the random walks process is described by parabolic equation with ordinary (*local*) Laplacian, and although every astrophysicist knew that interstellar medium (ISM) forms very inhomogeneous and strongly correlated on all scales system (*fractal*) [2–4], very few people point out the necessity to revise the transport equation.

Uchaikin and Gusarov [5], Uchaikin et al. [6] and Uchaikin and Korobko [7] have elaborated a model simulating fractal distribution in space. There has been found that the free path of a particle walking in this fractal system is distributed according to inverse power law. This served as a first spur to derive a new kinetic equation, integral (non-local) term of which includes the same inverse power type kernel, being nothing more than the isotropic Riesz–Feller fractional operator or the Laplace operator to power degree [8,9]. Significantly later, there was stated a link between this operator and turbulence spectrum of ISM [10].

Involvement of non-local operator $-(-\Delta)^{\alpha/2}$ instead of its local counterpart Δ has shown that the knee-effect arise in the en-

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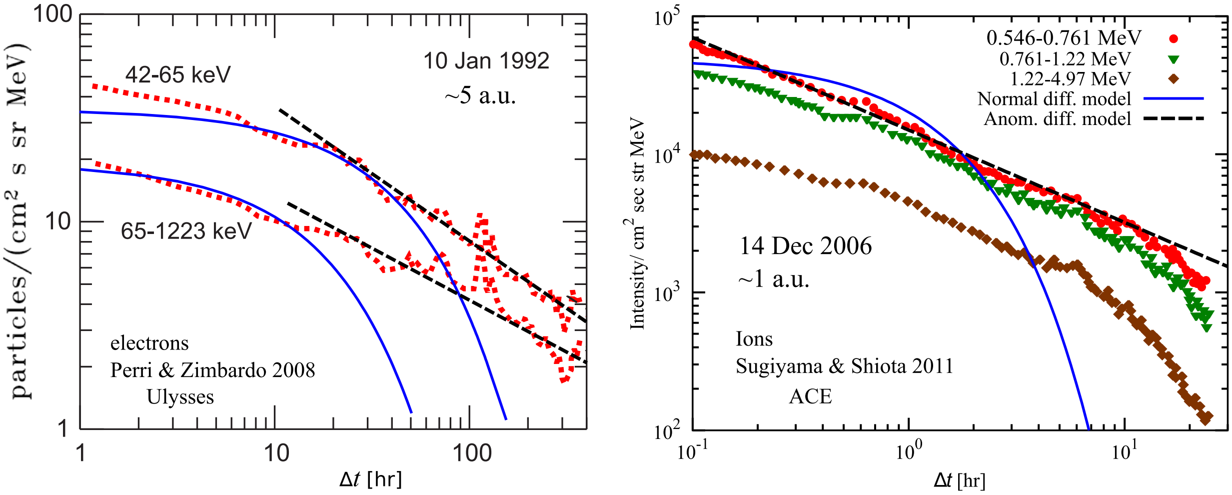

Fig. 1. Left panel: electron fluxes digitized from Ref. [11] reconstructed using Ulysses data (19 January 1992, ≈5 a.u.). Energy ranges are indicated. Blue solid lines are exponential plots corresponding to normal diffusion model. Dashed lines are power laws obtained by means of anomalous diffusion propagators (see details in [11]). Right panel: ion fluxes digitized from Ref. [23] reconstructed using ACE data (14 December 2006, ≈1 a.u.).

ergy spectrum from a local pulse source without any additional assumptions except the turbulent (called also *fractal*) structures, existence of which in the interstellar space is beyond doubt. During 2001–2010 years, a series of calculations was carried out by Lagutin et al. with the use of the nonlocal diffusion model (NoD-model), but all results were obtained without taking into account the relativistic limitation of speed and the boundary conditions.

In recent years, increasing evidence has been found that anomalous transport play an important role in CR propagation (e.g., [11–15]). It is known that anomalous diffusion is frequent behavior of transport of energetic charged particles in turbulent magnetic field (TMF). Chuvilgin and Ptuskin [16], Giacalone and Jokipii [17], Casse et al. [18] and Pucci et al. [19] predicted different transport regimes for parallel and perpendicular diffusion of charged particles in TMF among which are normal diffusion, ballistic motion, super- and subdiffusion. Realization of a particular regime depends on parameters, such as turbulence level $\delta B/B_0$, turbulence anisotropy, and ratio of Larmor radius R_L to correlation length l_c. Some experimental data [11,20] provide strong evidences of anomalous diffusion of cosmic rays in turbulent magnetic field. These data concern the case of solar CRs, but similar mechanism can lead to anomalous diffusion of galactic CRs. Perri and Zimbardo [11,20,21] analysed data sets of interplanetary shocks in the solar wind, the CIR-associated forward and reverse shocks detected by spacecrafts Ulysses and Voyager 2. They proposed the special procedure to compute fluxes of energetic particles by means of the propagator formalism and to determine type of diffusion from an observed particle flux.

To describe the superdiffusive regime, Perri and Zimbardo [11,20,21] used the power-law asymptotics obtained by Zumofen and Klafter [22] for Lévy walks – $G(x,t)=bt/x^{\alpha+1}$, ($b=\text{const}$, $1<\alpha<2$). Substituting approximate propagators and narrow source function into the formula for the particle time profile, they found agreement with the long-time region of the intensity of solar energetic particles. Sugiyama and Shiota [23] analyzing the data of December 14, 2006 from ACE (1 a.u.) confirmed in substance (Fig. 4, right panel) the observation of anomalous kind of solar CR diffusion (Fig. 1).

The question of the relativistic limitation of particle velocities also came rather late, one can say quite recently. Aloisio et al. [24] noted that “both diffusion equation and definition of the flux density do not know about light speed c and in fact result in superluminal motion... From the formal mathematical point of view the superluminal propagation exists always in the solutions of the non-relativistic diffusion equations, but very often the contribution of unphysical regions to the solution is negligibly small (e.g. in heat conductivity).” Diffusion coefficient of CR grows with energy and due to very wide spectra of CRs, mentioned contribution is not negligible for energies high enough. Prosekin et al. [25] provide the following comment about new relativistic diffusion model used by Aloisio et al. [24] “To avoid the problem of superluminal motion, Aloisio et al. [24] have introduced the so-called Jüttner function, which describes the evolution of the cosmic-ray density. Although this function is obtained phenomenologically from the formal similarity between the diffusion propagator and the Maxwellian distribution, it gives correct results in the limiting cases of diffusion and the ballistic regime. Below we will show that the Jüttner function integrated over time is close to our stationary solution, which proceeds from the Boltzmann equation.” They also notice that “while the diffusion of cosmic rays has been comprehensively studies in the literature, the description of propagation in the intermediate stage, i.e. at the transition from the ballistic to the diffusive regime, is a problem of greater complexity regarding the exact analytical solutions.”

Litvinenko et al. [26] also indicate the problem of the nonrelativistic diffusion approach to propagation of relativistic CRs: “A general shortcoming of the diffusion approximation is that the diffusion equation implies an infinite speed of signal propagation, whereas particle speeds are finite, of course. A more accurate description may be provided by the telegraph equation [27].” Although, the telegraph equation describes more accurately the one-dimensional random walk with finite velocity, but scattering points in space form an ensemble of independent (in probabilistic sense) points, like positions of molecules of a hypothetical ideal gas. In fact, the turbulent irregularities of the interstellar magnetic field introduce additional more powerful correlated fluctuations in the CR-transport process which qualitatively change CR trajectory character.

Implication of the speed limitation into the NoD-model was realized in [28,29] and later this version was named the *Nonlocal Relativistic Diffusion* (NoRD) model [30,31]. The physical interpretation of NoRD model and some results obtained in its frame were described in our reviews [32,10].

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