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Conditions for continuity of fractional velocity and existence of fractional Taylor expansions

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A B S T R A C T

Hölder functions represent mathematical models of nonlinear physical phenomena. This work investigates the general conditions of existence of fractional velocity as a localized generalization of ordinary derivative with regard to the exponent order. Fractional velocity is defined as the limit of the difference quotient of the function's increment and the difference of its argument raised to a fractional power. A relationship to the point-wise Hölder exponent of a function, its point-wise oscillation and the existence of fractional velocity is established. It is demonstrated that wherever the fractional velocity of non-integral order is continuous then it vanishes. The work further demonstrates the use of fractional velocity as a tool for characterization of the discontinuity set of the derivatives of functions thus providing a natural characterization of strongly non-linear local behavior. A link to fractional Taylor expansions using Caputo derivatives is demonstrated.

1. Introduction

Derivatives can be viewed as mathematical idealizations of the linear growth. Classical physical variables, such as velocity or acceleration, are considered to be differentiable functions of position. On the other hand, typical quantum mechanical paths [1–3] and Brownian motion trajectories were found to be non-differentiable.

Hence, mathematical descriptions of strongly non-linear phenomena necessitate certain relaxation of the linearity assumption. While this can be achieved in several ways, the present work focuses entirely on local descriptions in terms of limits of difference quotients. The theory is developed with the aim of providing tools for local study of strongly non-linear phenomena, for which ordinary derivatives diverge [4]. As useful conceptual models of such systems can be regarded singular functions, which violate the fundamental theorem of calculus.

The relaxation of the differentiability assumption opens new avenues in describing physical phenomena [5,6] but also challenges existing mathematical methods. Hölderian functions in this regard can be used as building blocks of such strongly non-linear models. Difference quotients of functions of fractional order have been considered for the first time by du Bois–Reymond [7] and Faber [8] in their studies of the point-wise differentiability of functions. While these initial development followed from purely mathematical interest later works were inspired from physical research questions related to fractal phenomena. Cherbit [9] and later on Ben Adda and Cresson [10] introduced the notion of *fractional velocity* as the limit of the fractional difference quotient. Existence of the fractional velocity was demonstrated for some classes of functions in [10–12].

The present work establishes further the general conditions of existence of fractional velocity. The main result of the paper is that for fractional orders the fractional velocity is continuous only if it is zero. The set of discontinuities of fractional velocity is characterized and used to describe the local change of the function in terms of its fractional Taylor expansion. In addition, the paper demonstrates that the regular Hölder functions are characterized by the fractional Taylor-Lagrange property; that is they can be approximated locally as fractional powers of appropriate orders.

In contrast to usual fractional derivative, the physical interpretation of fractional velocity is easier to establish due to its local

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character and the demonstrated fractional Taylor–Lagrange property (Eq. (4)). The fractional Taylor–Lagrange property was assumed and applied to establish a fractional conservation of mass formula in [13, Sec. 4] assuming the existence of a fractional Taylor expansion according to Odibat and Shawagfeh [14]. These authors derived fractional Taylor series development using repeated application of Caputo's fractional derivative[14]. A direct connection to Odibat and Shawagfeh's results is established in Section 9. In a related application, fractional velocities can be used to compute fractional Taylor expansions and to regularize derivatives of Hölder functions at non–differentiable points [15].

The results of the present paper are presented in a form accessible for an audience with diverse backgrounds, such as physics, computational biology, computer science etc.

2. General definitions and notations

The term *function* denotes a mapping $f : \mathbb{R} \mapsto \mathbb{R}$ (or implicitly $\mathbb{C} \mapsto \mathbb{C}$). The term *operator* denotes the mapping from functional expressions to functional expressions. Square brackets are used for the arguments of operators, while round brackets are used for the arguments of functions.

The term Cauchy sequence will be always interpreted as a null sequence.

Definition 1 (Asymptotic. $\mathcal{O}$ notation) The notation $\mathcal{O}(x^\alpha)$ is interpreted as the convention that

$$\lim_{x\to 0} \frac{\mathcal{O}(x^\alpha)}{x^\alpha} = 0$$

for $\alpha > 0$. The notation $\mathcal{O}(1)$ will be interpreted to indicate a Cauchy-null sequence.

Definition 2. We say that f is of (point-wise) Hölder class $\mathbb{H}^\beta$ if for a given x there exist two positive constants $C, \delta \in \mathbb{R}$ that for an arbitrary y in its domain and given $|x-y| \le \delta$ fulfill the inequality $|f(x) - f(y)| \le C|x-y|^\beta$, where $|\cdot|$ denotes the norm of the argument.

For (mixed) orders $n+\beta$ ($n \in \mathbb{N}_0$) the Hölder class $\mathbb{H}^{n+\beta}$ designates the functions for which the inequality

$$|f(x) - P_n(x-y)| \le C|x-y|^{n+\beta}$$

holds under the same hypothesis for C, δ and y. $P_n(z)$ designates the polynomial $P_n(z) = f(y) + \sum_{k=1}^{n} a_k z^k$.

Remark 1. The polynomial $P_n(x)$ can be identified with the Taylor polynomial of order n of $f(x)$ (see for example [11]).

Definition 3. Define the parametrized difference operators acting on the function $f(x)$ as

$$\Delta^+_\epsilon[f](x) := f(x+\epsilon) - f(x)\,,$$
$$\Delta^-_\epsilon[f](x) := f(x) - f(x-\epsilon)\,,$$
$$\Delta^2_\epsilon[f](x) := f(x+\epsilon) - 2f(x) + f(x-\epsilon)\,,$$

where $\epsilon > 0$[1]. The three operators are referred to as *forward difference, backward difference* and *second order difference* operators, respectively.

3. Point-wise oscillation of functions

The concept of point-wise oscillation is used to characterize the set of continuity of a function. To this end I build further on a technical result, which is presented as a Theorem 3.5.2 in Trench [16][p. 173]. Here the proof is slightly modified to account for separate treatment of right- and left- continuity.

Definition 4. Define forward oscillation and its limit as the operators

$$\mathrm{osc}^+_\epsilon[f](x) := \sup_{[x,x+\epsilon]}[f] - \inf_{[x,x+\epsilon]}[f]$$

$$\mathrm{osc}^+[f](x) := \lim_{\epsilon\to 0}\left(\sup_{[x,x+\epsilon]} - \inf_{[x,x+\epsilon]}\right) f = \lim_{\epsilon\to 0} \mathrm{osc}^+_\epsilon[f](x)$$

and backward oscillation and its limit as the operators

$$\mathrm{osc}^-_\epsilon[f](x) := \sup_{[x-\epsilon,x]}[f] - \inf_{[x-\epsilon,x]}[f]$$

$$\mathrm{osc}^-[f](x) := \lim_{\epsilon\to 0}\left(\sup_{[x-\epsilon,x]} - \inf_{[x-\epsilon,x]}\right) f = \lim_{\epsilon\to 0} \mathrm{osc}^-_\epsilon[f](x)$$

according to previously introduced notation [11].

Definition 5. The notation for the pair $\mu :: \epsilon$ will be interpreted as the implication that if LHS is fixed then RHS is fixed by the value chosen on the left, i.e. as an anonymous functional dependency $\epsilon = \epsilon(\mu)$.

Lemma 1 (Oscillation lemma)**.** *Consider the function $f : X \mapsto Y \subseteq \mathbb{R}$. Suppose that $I_+ = [x, x+\epsilon] \subseteq Dom[f]$, $I_- = [x-\epsilon, x] \subseteq Dom[f]$, respectively.*

If f is right-continuous in I_+ then $\mathrm{osc}^+[f](x) = 0$. Conversely, if $\mathrm{osc}^+[f](x) = 0$ then f is right-continuous in I_+.

If f is left-continuous in I_- then $\mathrm{osc}^-[f](x) = 0$. Conversely, if $\mathrm{osc}^-[f](x) = 0$ then f is left-continuous in I_-.

Proof.

Forward case Suppose that $\mathrm{osc}^+[f](x) = 0$. Then there exists a pair $\mu :: \delta, \delta \le \epsilon$, such that $\mathrm{osc}^+_\delta[f](x) \le \mu$. Therefore, f is bounded in I_+. Since μ is arbitrary we select x', such that

$$|f(x') - f(x)| = \mu' \le \mu$$

and set $|x - x'| = \delta'$. Since μ can be made arbitrary small so does μ'. Therefore, f is (right)-continuous at x.

Reverse case If f is (right-) continuous on x then there exist a pair $\mu :: \delta$ such that

$$\left|f(x') - f(x)\right| < \mu/2, \quad \left|x' - x\right| < \delta/2$$
$$\left|f(x) - f(x'')\right| < \mu/2, \quad \left|x - x''\right| < \delta/2$$

Then we add the inequalities and by the triangle inequality we have

$$\left|f(x') - f(x'')\right| \le \left|f(x') - f(x)\right| + \left|f(x) - f(x'')\right| < \mu$$
$$\left|x' - x''\right| \le \left|x' - x\right| + \left|x - x''\right| < \delta\,.$$

However, since x' and x'' are arbitrary we can set the former to correspond to the minimum and the latter to the maximum of f in the interval. therefore, by the least-upper-bond property we can identify $f(x') \mapsto \inf_\epsilon f(x)$, $f(x'') \mapsto \sup_\epsilon f(x)$. Therefore, $\mathrm{osc}^+_\delta[f](x) < \mu$ for $\left|x' - x''\right| < \delta$ (for the pair $\mu :: \delta$). Therefore, the limit is $\mathrm{osc}^+[f](x) = 0$.

The left case follows by applying the right case, just proved, to the mirrored image of the function: $f(-x)$. □

4. Fractional variations and fractional velocities

Definition 6. Define *fractional variation* operators of order $0 \le \beta \le 1$ as

$$\upsilon^{\epsilon+}_\beta[f](x) := \frac{\Delta^+_\epsilon[f](x)}{\epsilon^\beta} = \frac{f(x+\epsilon) - f(x)}{\epsilon^\beta} \qquad (1)$$

[1] assumed to hold throughout the paper for the variable ϵ.