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Multiplicity of solutions to fractional Hamiltonian systems with impulsive effects

Nemat Nyamoradi[a], Rosana Rodríguez-López[b,*]

[a] Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
[b] Departamento de Estadística, Análisis Matemático y Optimización, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782 Santiago de Compostela, Spain

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ABSTRACT

In this paper, we study the existence of infinitely many solutions to a class of boundary value problems for impulsive fractional Hamiltonian systems. The main tool is the use of variant Fountain theorems, which allow to give some sufficient conditions to guarantee that the impulsive problems object of our study have infinitely many solutions.

1. Introduction

The topic of fractional differential equations has gained increasing importance due to its applications in many different fields of engineering and sciences such as electricity, mechanics, biology, chemistry, control theory, signal and image processing, wave propagation, fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, probability, etc. For details on the applications to various fields, see, for instance, [1–7] and the references therein.

On the other hand, impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their state. Recent developments in this field have also been motivated by many problems corresponding to applications, such as control theory, population dynamics, medicine, etc. We refer to [8–17] for some monographs and papers including relevant information about this topic. For the application of critical point theory and variational methods for impulsive differential equations, we mention the pioneering works [18–20].

Combining both ideas, we get impulsive fractional differential equations, which have been studied in recent years from different points of view which allow to deduce the existence of solutions to this kind of equations (fixed point results, topological degree theory, upper and lower solutions method and monotone iterative technique,...), as can be seen, for instance, in [21–29] and the references therein. In particular, the authors of [23,26] analyze the existence of solutions to a boundary value problem of Dirichlet type for fractional differential equations subject to impulses, as follows,

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = \lambda f(t,u(t)), & t \neq t_j, \text{ a.e. } t \in [0,T], \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = \mu I_j(u(t_j)), & j = 1,2,\dots,n, \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where ${}_tD_T^{\alpha}$ and ${}_0^cD_t^{\alpha}$ are the right Riemann–Liouville and left Caputo fractional derivatives of order $\alpha \in (1/2, 1]$, respectively, $f \in C([0,T]\times\mathbb{R},\mathbb{R})$, $I_j \in C(\mathbb{R},\mathbb{R})$, $j = 1,\dots,n$, $a \in C([0,T],\mathbb{R})$ and λ, $\mu \in (0,+\infty)$ are two parameters. In those references, by using variational methods and critical point theory, it is deduced the existence of solutions to problem (1).

Motivated by the above facts, the aim of this paper is to establish the existence of infinitely many solutions to the following boundary value problem for impulsive fractional differential

* Corresponding author.
E-mail addresses: nyamoradi@razi.ac.ir, neamat80@yahoo.com (N. Nyamoradi), rosana.rodriguez.lopez@usc.es (R. Rodríguez-López).

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systems:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + A(t)u(t) = \nabla F(t,u(t)), & t \neq t_j, \ \text{a.e. } t \in [0,T], \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i))(t_j) = I_{ij}(u^i(t_j)), & i = 1,2,\dots,N, \\ & j = 1,2,\dots,l, \\ u(0) = u(T) = (0,\dots,0) \in \mathbb{R}^N, \end{cases} \tag{2}$$

where $\alpha \in (1/2, 1]$, $A : [0,T] \to \mathcal{M}_{N\times N}(\mathbb{R})$ is a continuous map from the interval [0, T] to the set of N-order symmetric matrices, $u(t) = (u^1(t), u^2(t), \dots, u^N(t))$, $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = T$, $I_{ij} : \mathbb{R} \to \mathbb{R}$, $i = 1,2,\dots,N$, $j = 1,2,\dots,l$, are continuous functions and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is such that the following assumption holds:

(**H**) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$ and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1([0,T], \mathbb{R}^+)$ such that

$$|F(t,x)| \le a(|x|)b(t), \qquad |\nabla F(t,x)| \le a(|x|)b(t),$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

In condition (H), $|\cdot|$ represents the p-norm in $\mathbb{R}^N$ $(1 < p < \infty)$ and, in problem (2), we denote

$$\Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i))(t_j) = {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i)(t_j^+) - {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i)(t_j^-),$$

$${}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i)(t_j^+) = \lim_{t\to t_j^+}({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i)(t)),$$

$${}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i)(t_j^-) = \lim_{t\to t_j^-}({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u^i)(t))$$

and $\nabla F(t,x) = (F_{x^1}(t,x), \dots, F_{x^N}(t,x))$, for a.e. $t \in [0, T]$ and all $x = (x^1, \dots, x^N) \in \mathbb{R}^N$.

In particular, if we take $\alpha = 1$, (2) reduces to the standard second order Hamiltonian system of the following form

$$\begin{cases} \ddot{u}(t) + A(t)u(t) = \nabla F(t,u(t)), & t \neq t_j, \ \text{a.e. } t \in [0,T], \\ \Delta(\dot{u}^i)(t_j) = I_{ij}(u^i(t_j)), & i = 1,2,\dots,N, \ j = 1,2,\dots,l, \\ u(0) = u(T) = (0,\dots,0) \in \mathbb{R}^N. \end{cases} \tag{3}$$

In both cases $A(t) \equiv 0$ and $A(t) \not\equiv 0$, the existence of solutions to the problem (3) has been intensively studied by many mathematicians (see, for instance, [30–33] and the references therein).

On the other hand, a very recent and relevant work on the homoclinic solutions for fractional Hamiltonian systems can be found in [34].

In this paper, we deal with the existence of solutions to problem (2). First, in Section 2, we give some preliminary definitions and recall some properties and results which are needed later. Then, in Section 3, we establish and prove our main results on the existence of solutions to problem (2).

2. Preliminaries on fractional calculus

To prove the main results, we need some preliminary notions and results which are recalled in this section. Here, we present briefly some basic notions corresponding to fractional calculus. The interested reader can find a detailed study in the monographs [35,36], as well as other texts on basic fractional calculus, while some of the properties in this section have been also taken from the paper [37].

Definition 1 (Left and right Riemann–Liouville fractional derivatives [35,36]). Let f be a function defined on [a, b]. The left and right Riemann-Liouville fractional derivatives of order $0 \le \gamma < 1$ for function f, denoted respectively by ${}_aD_t^{\gamma}f(t)$ and ${}_tD_b^{\gamma}f(t)$, are defined by

$${}_aD_t^{\gamma}f(t) = \frac{d}{dt}{}_aD_t^{\gamma-1}f(t)$$

$$= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\left(\int_a^t (t-s)^{-\gamma}f(s)ds\right), \quad t \in [a,b], \tag{4}$$

$${}_tD_b^{\gamma}f(t) = -\frac{d}{dt}{}_tD_b^{\gamma-1}f(t)$$

$$= -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\left(\int_t^b (s-t)^{-\gamma}f(s)ds\right), \quad t \in [a,b]. \tag{5}$$

Definition 2 (Left and right Caputo fractional derivatives [35]). Let $0 < \gamma < 1$ and $f \in AC([a, b])$, then the left and right Caputo fractional derivatives of order γ for function f denoted respectively by ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$, exist almost everywhere on [a, b]. The fractional derivatives ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$ are represented by

$${}_a^cD_t^{\gamma}f(t) = {}_aD_t^{\gamma-1}f'(t)$$

$$= \frac{1}{\Gamma(1-\gamma)}\left(\int_a^t (t-s)^{-\gamma}f'(s)ds\right), \quad t \in [a,b], \tag{6}$$

$${}_t^cD_b^{\gamma}f(t) = -{}_tD_b^{\gamma-1}f'(t)$$

$$= -\frac{1}{\Gamma(1-\gamma)}\left(\int_t^b (s-t)^{-\gamma}f'(s)ds\right), \quad t \in [a,b]. \tag{7}$$

It is known that, when $\gamma = 1$, ${}_a^cD_t^1f(t) = f'(t)$, ${}_t^cD_b^1f(t) = -f'(t)$, $t \in [a, b]$. For $\alpha = 0$, ${}_a^cD_t^0f(t) = {}_t^cD_b^0f(t) = f(t)$, $t \in [a, b]$.

Let $C_0^{\infty}([0,T])$ be the set of all functions $u \in C^{\infty}([0, T])$ with $u(0) = u(T) = 0$ and consider the norm $||u||_{\infty} = \max_{t\in[0,T]}|u(t)|$. We also consider the norm of the space $L^r([0, T])$, for $1 \le r < \infty$, given by

$$||u||_{L^r} = \left(\int_0^T |u(\xi)|^r d\xi\right)^{\frac{1}{r}}.$$

Definition 3 (Definition 3.1 [37]). Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^{\infty}([0,T])$, that is,

$$E_0^{\alpha,p} = \overline{C_0^{\infty}([0,T])}^{||\cdot||_{\alpha,p}}$$

where

$$||u||_{\alpha,p} = \left(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^cD_t^{\alpha}u(t)|^p dt\right)^{\frac{1}{p}}. \tag{8}$$

Lemma 1 (Proposition 3.1 [37]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

The following estimates are useful to our procedure and they refer to an equivalent norm in the space $E_0^{\alpha,p}$.

Lemma 2 (Proposition 3.2 [37]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, we have*

$$||u||_{L^p} \le \frac{T^{\alpha}}{\Gamma(\alpha+1)}||{}_0^cD_t^{\alpha}u||_{L^p}. \tag{9}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$||u||_{\infty} \le \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}||{}_0^cD_t^{\alpha}u||_{L^p}. \tag{10}$$

According to [37], taking into account (9), we can also consider $E_0^{\alpha,p}$ with respect to the equivalent norm

$$||u||_{\alpha,p} = ||{}_0^cD_t^{\alpha}u||_{L^p} = \left(\int_0^T |{}_0^cD_t^{\alpha}u(t)|^p dt\right)^{\frac{1}{p}}, \quad \forall\, u \in E_0^{\alpha,p}. \tag{11}$$

In the rest of the paper, we consider problem (2) in the context of the Hilbert space $X^{\alpha} := E_0^{\alpha,2}$, which can be equipped with the norm $||u||_{\alpha} = ||u||_{\alpha,2}$ defined in (11).

In what follows, we also assume that the coefficient mapping A satisfies the following conditions: