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A fractional Gauss–Jacobi quadrature rule for approximating fractional integrals and derivatives

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ABSTRACT

We introduce an efficient algorithm for computing fractional integrals and derivatives and apply it for solving problems of the calculus of variations of fractional order. The proposed approximations are particularly useful for solving fractional boundary value problems. As an application, we solve a special class of fractional Euler–Lagrange equations. The method is based on Hale and Townsend algorithm for finding the roots and weights of the fractional Gauss–Jacobi quadrature rule and the predictor-corrector method introduced by Diethelm for solving fractional differential equations. Illustrative examples show that the given method is more accurate than the one introduced in [26], which uses the Golub–Welsch algorithm for evaluating fractional directional integrals.

1. Introduction

Finding numerical approximations of fractional integrals or fractional derivatives of a given function is one of the most important problems in theory of numerical fractional calculus. The operators of fractional integration and fractional differentiation are more complicated than the classical ones, so their evaluation is also more difficult than the integer order case. Li et al. use spectral approximations for computing the fractional integral and the Liouville–Caputo derivative [22]. They also developed numerical algorithms to compute fractional integrals and Liouville–Caputo derivatives and for solving fractional differential equations based on piecewise polynomial interpolation [21]. In [29], Pooseh et al. presented two approximations derived from continuous expansions of Riemann–Liouville fractional derivatives into series involving integer order derivatives and they present application of such approximations to fractional differential equations and fractional problems of the calculus of variations. Some other computational algorithms are also introduced in [24,28,32]. For increasing the accuracy of the calculation, using the Gauss–Jacobi quadrature rule is appropriate for removing the singularity of the integrand. So considering the nodes and weights of the quadrature rule is an important problem. There are many good papers in the literature addressing the question of how to find the nodes and weights of the Gauss quadrature rule—see [6,36,37] and references therein. The more applicable and developed method is the Golub–Welsch (GW) algorithm [14,15], that is used by many of the mathematicians who work in numerical analysis. This method takes $O(n^2)$ operations to solve the problem of finding the nodes and weights. Here we use a new method introduced by Hale and Townsend [16], which is based on the Glasier–Liu–Rokhlin (GLR) algorithm [13]. It computes all the nodes and weights of the n-point quadrature rule in a total of $O(n)$ operations.

The structure of the paper is as follows. In Section 2, we introduce the definitions of fractional operators and some relations between them. Section 3 discusses the Gauss–Jacobi quadrature rule of fractional order and its application to approximate the fractional operators. In Section 4 we present two methods for finding the nodes and weights of Gauss–Jacobi and discuss their advantages and disadvantages. Two illustrative examples are solved. In Section 5 applications to ordinary fractional differential equations

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are presented. In Section 6 we investigate problems of the calculus of variations of fractional order and present a new algorithm for solving boundary value problems of fractional order. We end with Section 7 of conclusions and possible directions of future work.

2. Preliminaries and notations about fractional calculus

In this section we give some necessary preliminaries of the fractional calculus theory [18,27], which will be used throughout the paper.

Definition 2.1. The left and right Riemann–Liouville fractional integrals of order α of a given function f are defined by

$$ {}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt $$

and

$$ {}_xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, $$

respectively, where Γ is Euler's gamma function, that is,

$$ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, $$

$\alpha > 0$ with $n-1 < \alpha \le n$, $n \in \mathbb{N}$, and $a < x < b$. The left Riemann–Liouville fractional operator has the following properties:

$$ {}_aI_x^\alpha \, {}_aI_x^\beta = {}_aI_x^{\alpha+\beta}, \quad {}_aI_x^\alpha \, {}_aI_x^\beta = {}_aI_x^\beta \, {}_aI_x^\alpha, $$

$$ {}_aI_x^\alpha x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} x^{\alpha+\mu}, $$

where $\alpha, \beta \ge 0$ and $\mu > -1$. Similar relations hold for the right Riemann–Liouville fractional operator. On the other hand, we have the left and right Riemann–Liouville fractional derivatives of order $\alpha > 0$ that are defined by

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)dt $$

and

$$ {}_xD_b^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-\alpha-1} f(t)dt, $$

respectively.

There are some disadvantages when trying to model real world phenomena with fractional differential equations, when fractional derivatives are taken in Riemann–Liouville sense. One of them is that the Riemann–Liouville derivative of the constant function is not zero. Therefore, a modified definition of the fractional differential operator, which was first considered by Liouville and many decades later proposed by Caputo [7], is considered.

Definition 2.2. The left and right fractional differential operators in Liouville–Caputo sense are given by

$$ {}_a^CD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt $$

and

$$ {}_x^CD_b^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)dt, $$

respectively.

The Liouville–Caputo derivative has the following two properties for $n-1 < \alpha \le n$ and $f \in L_1[a, b]$:

$$ ({}_a^CD_x^\alpha \, {}_aI_x^\alpha f)(x) = f(x) $$

and

$$ ({}_aI_x^\alpha \, {}_a^CD_x^\alpha f)(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{(x-a)^k}{k!}, \quad t > 0. $$

Remark 2.3. Using the linearity property of the ordinary integral operator, one deduces that left and right Riemann–Liouville integrals, left and right Riemann–Liouville derivatives and left and right Liouville–Caputo derivatives are linear operators.

Another definition of a fractional differential operator, that is useful for numerical approximations, is the Grünwald–Letnikov derivative, which is a generalization of the ordinary derivative. It is defined as follows:

$$ D_{GL}^\alpha = \lim_{n\to\infty} \frac{(\frac{t}{N})^{-\alpha}}{\Gamma(-\alpha)} \sum_{j=0}^{n-1} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(t - \frac{tj}{n}\right). $$

3. Fractional Gauss–Jacobi quadrature rule

It is well known that the Jacobi polynomials $\{P_n^{(\lambda,\nu)}(x)\}_{n=0}^\infty$, $\lambda, \nu > -1$, $x \in [-1, 1]$, are the orthogonal system of polynomials with respect to the weight function

$$ (1-x)^\lambda (1+x)^\nu, \quad \lambda, \nu > -1, $$

on the segment $[-1, 1]$:

$$ \int_{-1}^1 (1-x)^\lambda (1+x)^\nu P_n^{(\lambda,\nu)}(x) P_m^{(\lambda,\nu)}(x) dx = \partial_n^{\lambda,\nu} \delta_{mn}, \tag{1} $$

where

$$ \partial_n^{\lambda,\nu} = \|P_n^{(\lambda,\nu)}\|^2_{w^{\lambda,\nu}}, \quad w^{\lambda,\nu}(x) = (1-x)^\lambda (1+x)^\nu, $$

$$ \delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \ne n \end{cases} $$

(see, e.g., [35]). These polynomials satisfy the three-term recurrence relation

$$ P_0^{(\lambda,\nu)}(x) = 1, \quad P_1^{(\lambda,\nu)}(x) = \frac{1}{2}(\lambda+\nu+2)x + \frac{1}{2}(\lambda-\nu), $$
$$ P_{n+1}^{(\lambda,\nu)}(x) = \left(a_n^{\lambda,\nu} x - b_n^{\lambda,\nu}\right) P_n^{(\lambda,\nu)}(x) - c_n^{\lambda,\nu} P_{n-1}^{(\lambda,\nu)}(x), \quad n \ge 2, \tag{2} $$

where

$$ a_n^{\lambda,\nu} = \frac{(2n+\lambda+\nu+1)(2n+\lambda+\nu+2)}{2(n+1)(n+\lambda+\nu+1)}, $$

$$ b_n^{\lambda,\nu} = \frac{(\nu^2-\lambda^2)(2n+\lambda+\nu+1)}{2(n+1)(n+\lambda+\nu+1)(2n+\lambda+\nu)}, $$

$$ c_n^{\lambda,\nu} = \frac{(n+\lambda)(n+\nu)(2n+\lambda+\nu+2)}{(n+1)(n+\lambda+\nu+1)(2n+\lambda+\nu)}. $$

The explicit form of the Jacobi polynomials is

$$ P_n^{(\lambda,\nu)}(x) = \sum_{k=0}^n \frac{2^{n-k} n! (k+\nu+1)_{n-k}}{(n-k)!(n+k+\lambda+\nu+1)_{n-k} k!} (t-1)^k, \tag{3} $$

where we use Pchhammer's notation:

$$ (a)_l = a(a+1)(a+2)\cdots(a+l-1) $$

(see [20]). Furthermore, the Jacobi polynomials satisfy in the following relations:

$$ P_n^{(\lambda,\nu)}(-x) = (-1)^n P_n^{(\lambda,\nu)}(x), $$

$$ \frac{d}{dx} P_n^{(\lambda,\nu)}(-x) = (-1)^n \frac{d}{dx} P_n^{(\lambda,\nu)}(x), $$